The average of cos is zero since phases are random so that

$$\langle I\rangle = |E_0|^2 N \tag{13.88}$$

(iv) The pulses can be made short by making sure that the extra phase of as many modes as possible are the same (known as mode-locking) and this would also lead to an increase in intensity which would then be proportional to N^2.

(6) We have that $\Delta kL = \pi/2$. Now, $\delta k = 2n_1 2\pi/10^{-6} - 2\pi n_2/5\times 10^{-5}$. Therefore, since $n_1 = 1.509$ and $n_2 = 1.530$, we have that $L = 1.210^{-5}$.

13.3 Problems and Solutions 3

13.3.1 *Problem set 3*

(1) Suppose that a two level system is in the state $a|0\rangle + b|1\rangle$. What are the probabilities of observing 0 and 1? What are the probabilities of observing $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$?

(2) A two state system with energy separation between the two states of $\hbar\omega_0$ is prepared in its ground state at $t = 0$. At $t = 0$ a sinusoidal perturbation

$$V(t) = \hbar V_0 \cos\omega t \tag{13.89}$$

is applied to the system (this assumes that there are only the off-diagonal elements, so that $V_{aa} = V_{bb} = 0$). Solving the time dependent Schrödinger equation in the two state basis and neglecting rapidly varying terms, show that the maximum probability of being in the excited state is given by

$$P_{\max} = \frac{|V_0|^2}{(\omega-\omega_0)^2 + |V_0|^2} \tag{13.90}$$

where $|V_0|$ is the matrix element of V_0 between the two states of the system. Show that a pulsed perturbation of duration $t = \pi/|V_0|$ with frequency $\omega = \omega_0$ inverts the system so that $p = 1$.

(3) A three level system interacting with an electromagnetic field can be described by the following Hamiltonian:

$$H_0 = \hbar\omega_a|a\rangle\langle a| + \hbar\omega_b|b\rangle\langle b| + \hbar\omega_c|c\rangle\langle c| \tag{13.91}$$

$$V(t) = -\frac{\hbar}{2}(\Omega_1 e^{-i(\phi_1+\nu_1)t}|a\rangle\langle b| + \Omega_2 e^{-i(\phi_2+\nu_2)t}|a\rangle\langle c|) + \text{h.c.}$$

Write down the total Hamiltonian in the matrix form. Suppose first that there is no interaction. What is the free evolution of the system? Suppose then that the interaction is turned on and that $\phi_1 = -\nu_1$ and $\phi_2 = -\nu_2$. Find the eigenvectors of the resulting interaction Hamiltonian.

(4) Explain in detail the semi-classical model of light matter interactions. Name one deficiency of this model and explain your choice.
A non-interacting two level atom is described by the following Hamiltonian:

$$H_0 = E_1|1\rangle\langle 1| + E_2|2\rangle\langle 2| \tag{13.92}$$

Explain the physical meaning of each term in this expression. This atom then interacts with the electromagnetic field such that the interaction Hamiltonian is given by

$$V(t) = \gamma e^{i\omega t}|1\rangle\langle 2| + \gamma e^{-i\omega t}|1\rangle\langle 2| \tag{13.93}$$

Explain the physical meaning of every term in the interaction Hamiltonian.
Assume that the atom is initially in the state 1. By knowing that the state of the atom is in general given by

$$c_1(t)e^{-iE_1t}|1\rangle + c_2(t)e^{-iE_1t}|1\rangle \tag{13.94}$$

solve the Schrödinger equation to show that the probability of occupying the exited state oscillates at the frequency

$$\Omega = \sqrt{\gamma^2/\hbar^2 + (\omega - \omega_{12})^2/4)} \tag{13.95}$$

i.e.

$$|c_2(t)|^2 \propto \sin^2 \Omega t \tag{13.96}$$

where

$$\omega_{12} = \frac{E_2 - E_1}{\hbar} \tag{13.97}$$

(Do not derive the constant of proportionality.)
Assume that the interaction is "on-resonant". Derive and plot the exact evolution of the probability to occupy the excited state 2 as a function of time. Given that $\gamma = 10^{-24}J$, how long does it take to excite the atom to level 2 if it is initially in level 1?

(5) Explain briefly the semi-classical approximation in the treatment of light matter interactions.
A nuclear spin has two possible states in a external magnetic field, up $|\uparrow\rangle$ (i.e. aligned with the field) and down $|\downarrow\rangle$ (i.e. anti-aligned with the field). Suppose that the nucleus is in an external (static) magnetic field of strength B, which points in the z direction.

(i) Write down the Hamiltonian for the nucleus using the Pauli matrix notation and identify its eigenvalues.
(ii) Suppose that the initial state of the system is aligned with the field in the z direction, $|\uparrow\rangle$. Suppose then that the field is instantaneously switched to the x direction. Solve the Schrödinger equation to obtain the exact evolution of the nuclear spin in terms of the eigenstate of the Pauli spin matrix σ_x. What is the phase difference between the two orthogonal spin eigenstates of σ_x as a function of time?
(iii) After what time will the spin switch to *its* orthogonal state $|\downarrow\rangle$?

What is the relationship between the energy associated with the spin and the time it takes to evolve between orthogonal states? Comment on the validity of the time energy uncertainty relation to estimate the time of this transition.
The Pauli matrices are given in the $|\uparrow\rangle$, $|\downarrow\rangle$ basis by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{13.98}$$

13.3.2 *Solutions 3*

(1) The probabilities in the 0, 1 basis are $|a|^2$ and $|b|^2$, while in the $\pm$ basis they are $|a+b|^2$ and $|a-b|^2$.
(2) Let's start by writing

$$\Psi = c_1(t)e^{-iE_1t/\hbar}|1\rangle + c_2(t)e^{-iE_2t/\hbar}|2\rangle \tag{13.99}$$

Substituting this into the Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = (H_0 + \hbar V(t))\Psi \tag{13.100}$$

we obtain

$$\dot{c}_1 = -i\bar{V}_0 \cos\omega t e^{-i\omega_0 t} c_2 \tag{13.101}$$
$$\dot{c}_2 = -i\bar{V}_0^* \cos\omega t e^{i\omega_0 t} c_1 \tag{13.102}$$

After applying the rotating wave approximation we obtain

$$\dot{c}_1 = -i\bar{V}_0 e^{i(\omega-\omega_0)t} c_2 \tag{13.103}$$

$$\dot{c}_2 = -i\bar{V}_0^* e^{-i(\omega-\omega_0)t} c_1 \tag{13.104}$$

The initial conditions are $c_1(0) = 1, c_2(0) = 0$ and $\dot{c}_2(0) = -i/2\bar{V}_0^*, \dot{c}_1(0) = 0$. Eliminating c_1 by double differentiation we arrive at

$$\ddot{c}_2 + i(\omega - \omega_0)\dot{c}_2 + \frac{1}{4}|\bar{V}_0|^2 c_2 = 0 \tag{13.105}$$

A trial solution is $e^{i\mu t}$ and the resulting equation is

$$-\mu^2 - (\omega - \omega_0)\mu + \frac{1}{4}|\bar{V}_0|^2 = 0 \tag{13.106}$$

The roots are

$$\mu_\pm = \frac{1}{2}(-(\omega - \omega_0) \pm [(\omega - \omega_0)^2 + |\bar{V}_0|^2]^{1/2}) \tag{13.107}$$

and the full solution to the Schrödinger equation is thus

$$c_2(t) = A_+ e^{i\mu_+ t} + A_- e^{i\mu_- t} \tag{13.108}$$

The coefficients $A_\pm$ can be fixed from the initial conditions to yield

$$c_2(t) = -\frac{|\bar{V}_0|^2}{2} \frac{1}{(\omega - \omega_0)^2 + |\bar{V}_0|^2} (e^{i\mu_+ t} - e^{i\mu_- t}) \tag{13.109}$$

where

$$\Omega = (\omega - \omega_0)^2 + |\bar{V}_0|^2]^{1/2} \tag{13.110}$$

is the Rabi frequency. Thus the solution for the upper level evolution is

$$c_2(t) = -\frac{i}{\Omega} \bar{V}_0^* e^{-i(\omega-\omega_0)t/2} \sin \frac{1}{2}\Omega t \tag{13.111}$$

The probability in question now follows from this. The flip is reached when $\sin \frac{1}{2}\Omega t = 1$, which results in the π-pulse of duration π/Ω.

(3) Solve the Schrödinger equation to obtain the free evolution. The solution should be of the form

$$|\Psi(t)\rangle = c_a e^{-i\omega_a t}|a\rangle + c_b e^{-i\omega_b t}|b\rangle + c_c e^{-i\omega_c t}|c\rangle \tag{13.112}$$

where c_a, c_b, c_c are initial amplitudes for states a, b, c respectively. When you diagonalize the interaction Hamiltonian you should obtain the following eigenstates:

$$|\psi_\pm\rangle = \frac{1}{\sqrt{2}}\left(|a\rangle \pm \frac{\Omega_1}{\Omega}|b\rangle \pm \frac{\Omega_2}{\Omega}|c\rangle\right) \tag{13.113}$$

$$|\psi_0\rangle = \left(\frac{\Omega_2}{\Omega}|b\rangle - \frac{\Omega_1}{\Omega}|c\rangle\right) \tag{13.114}$$

where $\Omega = \sqrt{\Omega_1^2 + \Omega_2^2}$. (What are the corresponding eigenvalues?).

(4) In the semi-classical model atoms are quantized, but light is not. The atom is described through a Hamiltonian, and the effect of light is taken as an additional part of the Hamiltonian. The evolution of the system is obtained by solving the Schrödinger equation with the total Hamiltonian. This is frequently impossible to solve analytically and we have to resort to approximations or numerics. The effect of oscillating fields is usually taken as a perturbation of the basic non-interacting atomic Hamiltonian. This leads to the time dependent perturbation theory where the most useful result is Fermi's golden rule. This tells us the probability of obtaining a transition from one level to another under a time dependent perturbation.
$|1\rangle$ and $|2\rangle$ represent the two atomic levels. The E_1 and E_2 are the corresponding energies of the two states. They are the eigenvalues of the atomic Hamiltonian with $|1\rangle$ and $|2\rangle$ being the eigenvectors. Now, when this atom interacts with a field the Hamiltonian contains the transition elements for jumping from 1 to 2 and vice versa. We have the creation and annihilation operators $|2\rangle\langle 1|$ and $|1\rangle\langle 2|$.
The Schrödinger equation is

$$\begin{aligned}&(E_1|1\rangle\langle 1| + E_2|2\rangle\langle 2| + \gamma e^{i\omega t}|1\rangle\langle 2| + \gamma e^{-i\omega t}|2\rangle\langle 1|)|\Psi(t)\rangle \\ &= i\hbar\frac{\partial|\Psi(t)\rangle}{\partial t}\end{aligned} \tag{13.115}$$

By substituting in the wave function

$$|\Psi(t)\rangle = c_1(t)e^{-iE_1t}|1\rangle + c_2(t)e^{-iE_2t}|2\rangle \tag{13.116}$$

we obtain two equations:

$$\frac{\gamma}{\hbar}c_2 = i\dot{c}_1 e^{i(\omega_{12}-\omega)t} \tag{13.117}$$

$$\frac{\gamma}{\hbar}c_1 = i\dot{c}_2 e^{-i(\omega_{12}-\omega)t} \tag{13.118}$$

This is a system of coupled equations which we solve for c_2 by differentiating the second equation and substituting the first equation into it. We obtain

$$\ddot{c}_2 - i(\omega_{12} - \omega)\dot{c}_2 + \frac{\gamma^2}{\hbar^2}c_2 = 0 \tag{13.119}$$

The trial solution $c_2 = e^{i\mu t}$ leads to

$$\mu^2 - (\omega_{12} - \omega)\mu - \frac{\gamma^2}{\hbar^2} = 0 \tag{13.120}$$

which has two roots:

$$\mu_{1,2} = \frac{\omega_{12} - \omega}{2} \pm (\sqrt{\gamma^2/\hbar^2 + (\omega - \omega_{12})^2/4}) \tag{13.121}$$

Therefore

$$c_2(t) = Ae^{i\mu_1 t} + Be^{i\mu_2 t} \tag{13.122}$$

But $c_2(0) = 0$, thus $A = -B$. The solution is therefore

$$|c_2|^2 = 4A^2 \sin^2(\Omega t) \tag{13.123}$$

as required.
If on resonance, we have

$$|c_2|^2 = \sin^2(\gamma t/\hbar) \tag{13.124}$$

For $|c_2|^2 = 1$ we require

$$\frac{\gamma t}{\hbar} = \frac{\pi}{2} \tag{13.125}$$

and so we obtain the time for a flop to be $t = \pi\hbar/2\gamma$. Given that $\gamma = 10^{-24}$J, we find $t = 1.65 \times 10^{-10}$ seconds.

(5) The first part is the same as the previous question.

(i) The initial Hamiltonian is $H = \frac{\mu B}{2}\sigma_z$. This has eigenvalues $-\mu B/2$ and $+\mu B/2$ as can be seen from the matrix form:

$$H = \begin{pmatrix} \frac{\mu B}{2} & 0 \\ 0 & -\frac{\mu B}{2} \end{pmatrix}$$

(ii) Now the state is $|\uparrow\rangle$, but the Hamiltonian is $H = \frac{\mu B}{2}\sigma_x$. The eigenvectors of this Hamiltonian are $|\rightarrow\rangle$ and $|\leftarrow\rangle$, which evolve with phases $e^{-i\mu Bt/2\hbar}$ and $e^{i\mu Bt/2\hbar}$ respectively. But, the initial state can be written as an equal superposition of the eigenstates of σ_x:

$$|\uparrow\rangle = |\leftarrow\rangle + |\rightarrow\rangle \tag{13.126}$$

Therefore the state after some time T is given by

$$|\psi(T)\rangle = e^{i\mu BT/2\hbar}|\leftarrow\rangle + e^{-i\mu BT/2\hbar}|\rightarrow\rangle \tag{13.127}$$

The phase difference between the two states is

$$\Delta\phi = \frac{\mu BT}{\hbar} \tag{13.128}$$

(3) The orthogonal state to the initial state is

$$|\downarrow\rangle = |\leftarrow\rangle - |\rightarrow\rangle \tag{13.129}$$

So, the phase difference between the states has to be $e^{i\mu BT/\hbar} = -1$ and so

$$T = \frac{\pi\hbar}{\mu B} \tag{13.130}$$

The energy difference is $E = \mu B$ and so from the last question we have that

$$TE = \pi\hbar \tag{13.131}$$

This is very similar to the uncertainty relation between energy and time $\Delta E\Delta t \geq \hbar$.
Thus the time energy uncertainty is a pretty good estimate of the time it takes for the transition, given that it is only out by π.

13.4 Problems and Solutions 4

13.4.1 *Problem set 4*

(1) In the Schrödinger picture, operators are time independent and states evolve according to the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\Psi\rangle = \hat{H}|\Psi\rangle \tag{13.132}$$

where $\hat{H}$ is the (time independent) Hamiltonian. Find a form for the time evolution operator

$$\hat{U}(t, t_0)|\Psi(t_0)\rangle = |\Psi(t)\rangle \tag{13.133}$$

in terms of the Hamiltonian $\hat{H}$. Use this to show that in the Heisenberg picture, where states do not evolve in time but operators representing observables do, that

$$i\hbar\frac{d}{dt}\hat{O}(t) = [\hat{O}, \hat{H}] \tag{13.134}$$

(2) Consider a two level atom with the ground state $|g\rangle$ and excited state $|e\rangle$ interacting with a single quantized radiation field mode of frequency ω which is close (but not equal) to the atomic transition frequency ω_0. The interaction energy between the atomic dipole $\hat{d}$ and the electric field $\hat{E}$ is $\hat{V} = -\hat{d}\hat{E}$. We write the dipole operator as $\hat{d} = d_{eg}(\sigma_+ + \sigma_-)$ where $\sigma_\pm$ are the Pauli raising and lowering operators

$$\hat{\sigma}_+|g\rangle = |e\rangle \tag{13.135}$$

$$\hat{\sigma}_-|e\rangle = |g\rangle \tag{13.136}$$

so $\hat{\sigma}_+ = |e\rangle\langle g|$. The field operator $\hat{E} = E_0(\hat{a} + \hat{a}^\dagger)\sin kz$, where $\hat{a}$ and $\hat{a}^\dagger$ are the creation and annihilation operators. Use the result of question 1 to show that the rotating wave approximation of neglecting fast varying non-resonant coupling is equivalent to dropping terms $\hat{a}^\dagger\hat{\sigma}_- + \hat{a}\hat{\sigma}_+$ in V.

(3) A single mode field is prepared in a number state $|n\rangle$ with precisely n photons. Calculate the uncertainty in the field operator

$$\hat{E} = E_0(\hat{a} + \hat{a}^\dagger)\sin kz \tag{13.137}$$

and interpret the result.

(4) Explain the process of quantizing the electromagnetic field. Why is a single mode of the field equivalent to a unit mass harmonic oscillator?
In terms of the creation and annihilation operators, the Hamiltonian for a single mode field of frequency ω is

$$H = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \tag{13.138}$$

A coherent state of this field with the amplitude α is given by

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{13.139}$$

What is the probability of obtaining n photons in the field? What is the average energy of the field in this state? Hence give the physical meaning of the amplitude α.
Solve the Schrödinger equation to obtain the free evolution of this state. What happens to the amplitude α during the evolution?

The action of a beam splitter with the transmission and reflection amplitudes T and R respectively is given by

$$|n\rangle \otimes |0\rangle \to \sum_p T^{n-p} R^p \sqrt{\binom{n}{p}} |n-p\rangle \otimes |p\rangle \qquad (13.140)$$

Suppose that the input is a coherent state of amplitude α in one port and vacuum in the other port, $|\alpha\rangle \otimes |0\rangle$. Show that the output is a product of coherent states

$$|T\alpha\rangle \otimes |R\alpha\rangle \qquad (13.141)$$

Hence explain why the coherent state is considered the best quantum description of classical light.

(5) A quantum particle moving non-relativistically in one dimension has mass m and potential energy $\frac{1}{2}m\omega^2x^2$. Write down its Hamiltonian H. Express H in terms of the operators

$$a = \frac{\beta}{\sqrt{2}}\left(x + i\frac{p}{m\omega}\right) \qquad (13.142)$$

$$a^\dagger = \frac{\beta}{\sqrt{2}}\left(x - i\frac{p}{m\omega}\right) \qquad (13.143)$$

where $\beta^2 = m\omega/\hbar$. You may assume that $[x, p] = i\hbar$.

(i) Evaluate the commutators $[a, a^\dagger]$, $[H, a^\dagger]$ and $[H, a]$.

(ii) Hence determine the allowed energy levels of the particle, explaining carefully the logic that you use. What do these levels represent when we apply them to a single mode of the quantized electromagnetic field?

(iii) Let $|0\rangle$ denote the ground state. Show that

$$\langle 0|(a + a^\dagger)|0\rangle = 0 \qquad (13.144)$$

$$\langle 0|(a + a^\dagger)^2|0\rangle = 1 \qquad (13.145)$$

What do these relationships signify in relation to the quantized electromagnetic field?

(6) Suppose that we have a state of two light modes of the form $1/2|2,0\rangle + 1/\sqrt{2}|1,1\rangle + 1/2|2,0\rangle$. Prove that this state is entangled, i.e. prove that the state cannot be written as a state of light $(a|0\rangle + b|1\rangle + c|2\rangle)(e|0\rangle + f|1\rangle + g|2\rangle)$.

13.4.2 *Solutions 4*

(1) From the Schrödinger equation it follows that

$$|\Psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\Psi(0)\rangle \qquad (13.146)$$

(Note that this is only true if the Hamiltonian is time independent.) Therefore

$$|\Psi(t)\rangle = e^{-i\hat{H}(t-t_0)/\hbar}|\Psi(t_0)\rangle \tag{13.147}$$

and so $\hat{U}(t,t_0) = e^{-i\hat{H}(t-t_0)/\hbar}$ is the time evolution operator. The usual expectation value of an observable is given by

$$\langle \hat{O}(t)\rangle = \langle \Psi(t)|\hat{O}|\Psi(t)\rangle \tag{13.148}$$

But this can be rewritten using our time evolution operator as

$$\langle \hat{O}(t)\rangle = \langle \Psi(0)|e^{i\hat{H}(t-t_0)/\hbar}\hat{O}e^{-i\hat{H}(t-t_0)/\hbar}|\Psi(0)\rangle \tag{13.149}$$

and so

$$\hat{O}(t,t_0) = e^{i\hat{H}(t-t_0)/\hbar}\hat{O}e^{-i\hat{H}(t-t_0)/\hbar} \tag{13.150}$$

is the time dependent operator in the Heisenberg picture. Let's now derive the evolution equation for the above operator. Differentiate the equation obtaining

$$i\hbar\frac{d}{dt}\hat{O}(t,t_0) = -\hat{H}\hat{O}(t) + \hat{O}(t)\hat{H} = [\hat{O}(t),\hat{H}] \tag{13.151}$$

(2) Using the previous question we have

$$i\hbar\frac{d}{dt}\sigma_+ = [\hat{\sigma}_+,\hat{H}] \tag{13.152}$$

where $\hat{H} = \hat{H}_{\text{atom}} + \hat{H}_{\text{field}}$ and $[\hat{\sigma},\hat{H}_F] = 0$. The trick here is to identify the Hamiltonian for the atom. It is most conveniently given by

$$\hat{H}_{\text{atom}} = E_e|e\rangle\langle e| + E_g|g\rangle\langle g| \tag{13.153}$$

Let's assume for simplicity that $E_g = 0$. Then $E_e = \hbar\omega_0$ which is the transition energy. So

$$\frac{d}{dt}\sigma_+ = -\frac{i}{\hbar}[\sigma_+, \hbar\omega_0|e\rangle\langle e|] \tag{13.154}$$

$$= -\frac{i}{\hbar}(\hbar\omega_0[|e\rangle\langle g|e\rangle\langle e| - |e\rangle\langle e|e\rangle\langle g|]) \tag{13.155}$$

so that

$$\frac{d}{dt}\sigma_+ = \frac{i}{\hbar}(\hbar\omega_0)\sigma_+ \tag{13.156}$$

$$\sigma_+(t) = e^{i\omega_0 t}\sigma_+(0) \tag{13.157}$$

Similarly,

$$\sigma_-(t) = e^{-i\omega_0 t}\sigma_-(0) \tag{13.158}$$

Now we do the same for the field equations. We use $\hat{H}_F = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2)$ and $[\hat{a}, \hat{a}^\dagger] = 1$. Then

$$i\hbar\frac{d}{dt}\hat{a}(t) = [\hat{a}(t), \hat{H}] = [\hat{a}(t), \hat{H}_F] \tag{13.159}$$

is the evolution equation which yields (prove it!)

$$\hat{a}(t) = \hat{a}(0)e^{-i\omega t} \tag{13.160}$$
$$\hat{a}^\dagger(t) = \hat{a}^\dagger(0)e^{i\omega t} \tag{13.161}$$

So we can write the interaction as

$$\hat{V}(t) = -\hat{d}\hat{E}(t) = -d_{eg}(\hat{\sigma}_+ + \hat{\sigma}_-)E_0 \sin kz(\hat{a}^\dagger(t) + \hat{a}(t)) \tag{13.162}$$

Terms $\sigma_+ a \sim e^{i(\omega-\omega_0)t}$ and $\sigma_- a^\dagger \sim e^{-i(\omega-\omega_0)t}$ are slowly varying near resonance, but $\sigma_+ a^\dagger \sim e^{i(\omega+\omega_0)t}$ and $\sigma_- a \sim e^{-i(\omega+\omega_0)t}$ very at $\sim 2\omega_0$ and average to zero, leading to the rotating wave approximation.

(3) The mean field $\langle E\rangle$ is zero in a number state, as

$$\langle n|a + a^\dagger|n\rangle = 0 \tag{13.163}$$

On the other hand, the mean square field is

$$\begin{aligned}\langle n|E^2|n\rangle &= E_0^2 \sin^2 kz\langle n|a^2 + a^\dagger a + aa^\dagger + (a^\dagger)^2|n\rangle \\ &= E_0^2 \sin^2 kz(2n+1)\end{aligned} \tag{13.164}$$

The field with fixed n has definite amplitude, but random phase between 0 and 2π.

(4) The electromagnetic field is classically a wave described by six numbers at every point in space and time. These numbers can be specified simultaneously and the values of both the electric and magnetic field can in principle be determined exactly and simultaneously. When the field is quantized this is no longer possible. In fact, the electric and magnetic fields become operators which are no longer commuting, i.e. they are no longer simultaneously measurable. Writing the total classical energy in the field we get

$$W = \frac{1}{2}\int(\epsilon E^2 + \mu H^2)dV \tag{13.165}$$

But $E \propto x\cos\omega t/V$ and $B \propto p\omega \sin\omega t/V$, so that $W = (p^2 + \omega^2 x^2)/2$, which is a simple harmonic oscillator.

In terms of the creation and annihilation operators, the Hamiltonian for a single mode field of frequency ω is

$$H = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \tag{13.166}$$

A coherent state of this field with the amplitude α is given by

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{13.167}$$

What is the probability of obtaining n photons in the field?
The probability of obtaining n photons is $|\langle n|\alpha\rangle|^2$ and is

$$p_n = e^{-|\alpha|^2}\frac{|\alpha|^2 n}{n!} \tag{13.168}$$

The average energy is

$$\langle E\rangle = \hbar\omega\langle\alpha|\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|\alpha\rangle \tag{13.169}$$

$$= \hbar\omega\left(e^{-|\alpha|^2}\sum_n \frac{|\alpha|^{2n}}{n!}n + \frac{1}{2}\right) \tag{13.170}$$

$$= \hbar\omega\left(e^{-|\alpha|^2}|\alpha|^2\sum_n \frac{|\alpha|^{2(n-1)}}{(n-1)!} + \frac{1}{2}\right) \tag{13.171}$$

$$= \hbar\omega\left(|\alpha|^2 + \frac{1}{2}\right) \tag{13.172}$$

The physical meaning of $|\alpha|^2$ is that it is the average number of photons.
We now solve the Schrödinger equation to obtain the free evolution of this state,

$$i\hbar\frac{\partial|n\rangle}{\partial t} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|n\rangle \tag{13.173}$$

The solution is

$$|n(t)\rangle = e^{-i\omega(n+1/2)t}|n\rangle \tag{13.174}$$

Therefore

$$|\alpha(t)\rangle = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}e^{-i\omega(n+1/2)t}|n\rangle \tag{13.175}$$

$$= e^{-i\omega t/2}e^{-|\alpha|^2/2}\sum_n \frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}}|n\rangle \tag{13.176}$$

$$= e^{-i\omega t/2}|\alpha e^{-i\omega t}\rangle \tag{13.177}$$

Therefore the amplitude oscillates at frequency ω.
When the coherent state is the input we have

$$|\alpha\rangle \otimes |0\rangle \to e^{-|\alpha|^2/2} \sum_n \sum_p T^{n-p} R^p \sqrt{\binom{n}{p}} \frac{\alpha^n}{\sqrt{n!}} |n-p\rangle \otimes |p\rangle$$

$$= e^{-|T\alpha|^2/2} e^{-|R\alpha|^2/2} \sum_n \sum_p \left\{ \frac{T\alpha^{n-p}}{\sqrt{(n-p)!}} \right\} |n-p\rangle$$

$$\otimes \left\{ \frac{R\alpha^p}{\sqrt{p!}} \right\} |p\rangle$$

$$= |T\alpha\rangle \otimes |R\alpha\rangle \tag{13.178}$$

QED.

(5) The Hamiltonian is given by

$$H = \left(\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \right) \tag{13.179}$$

We can first compute

$$a^\dagger a = \frac{\beta^2}{2} \left(x^2 + \frac{i}{m\omega}[x,p] + \frac{p^2}{m\omega} \right) \tag{13.180}$$

Using the fact that $[x,p] = i\hbar$ we can rearrange the above to obtain

$$H = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right) \tag{13.181}$$

(i) We can easily prove that $[a, a^\dagger] = 1$, from which it follows that

$$[H, a] = \hbar\omega [a^\dagger, a] a = -\hbar\omega a \tag{13.182}$$

and

$$[H, a^\dagger] = \hbar\omega a^\dagger [a, a^\dagger] = \hbar\omega a^\dagger \tag{13.183}$$

(ii) Suppose that $H|\psi_n\rangle = E_n|\psi_n\rangle$. Then $(Ha - aH)|\psi_n\rangle = (H - E_n)a|\psi_n\rangle = -\hbar\omega a|\psi_n\rangle$. Thus,

$$H(a|\psi_n\rangle) = (E_n - \hbar\omega)(a|\psi_n\rangle) \tag{13.184}$$

Thus, a lowers energy by $\hbar\omega$ and, likewise, it can be shown that $a^\dagger$ raises energy by $\hbar\omega$.
The lowest energy state is the one which cannot be lowered anymore, so

$$a|0\rangle = 0 \tag{13.185}$$

Therefore, $H|0\rangle = \hbar\omega/2|0\rangle$ and so the energy of the nth eigenstate is $E_n = (n+1/2)\hbar\omega$. When we quantize the electromagnetic field, then each mode (frequency in other words) is a harmonic oscillator. Different levels represent different number of photons that can exist at the given frequency (or with the corresponding energy).

(iii) The first equality is easy as $a|0\rangle = 0$ and likewise $\langle 0|a^\dagger = 0$. The second is done by expansion, since

$$(a+a^\dagger)^2 = a^2 + aa^\dagger + a^\dagger a + (a^\dagger)^2 \tag{13.186}$$

All terms disappear apart from

$$\langle 0|aa^\dagger|0\rangle = 1 \tag{13.187}$$

The first relationship says that the average of the electric field in the vacuum is zero. The second says that the average of the energy squared is non-zero. This is a purely quantum signature, and (loosely speaking) says that the vacuum state also contains some energy, since energy of the field is proportional to its square.

(6) It is clear that there are no coefficients a, b, c, d, e, f and g that can reproduce the original state. Therefore the state cannot be written as a product and this means that it is entangled.

13.5 Problems and Solutions 5

13.5.1 *Problem set 5*

(1) The coherent state $|\alpha\rangle$ is generated from the vacuum state of a field mode $|0\rangle$ by a unitary transformation:

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle \tag{13.188}$$

where $\hat{D}(\alpha) = e^{+\alpha a^\dagger} e^{-|\alpha|^2/2}$ is the Glauber displacement operator, $\hat{a}^\dagger$ the creation operator. Show that, using this definition, the probability of finding n photons in a coherent state $|\alpha\rangle$, $p(n) = |\langle n|\alpha\rangle|^2$ is given by the Poisson formula

$$p(n) = \frac{\bar{n}^n e^{-\bar{n}}}{n!} \tag{13.189}$$

and determine the mean number of photons $\bar{n}$ and the standard deviation $\Delta n = [\langle n^2\rangle - \langle n\rangle^2]^{1/2}$.

Calculate the mean field $\langle\hat{E}\rangle$ for an electric field operator $\hat{E} = E_0 \sin kz(\hat{a} + \hat{a}^\dagger)$ for a field prepared in a number state. Then calculate the mean square field $\langle\hat{E}^2\rangle$ in such a coherent state, and from these results find the field uncertainty $\Delta E = [\langle \hat{E}^2 \rangle - \langle \hat{E} \rangle^2]^{1/2}$. How does this compare with the field uncertainty for the vacuum?

(2) Imagine we can prepare a single field mode in a superposition of zero and ten photons

$$|\Psi\rangle = a|0\rangle + b|10\rangle \tag{13.190}$$

Take $a = b = 1/\sqrt{2}$ for simplicity. Calculate the mean photon number for this superposition. Suppose having prepared initially such a superposition state, we monitor the field and detect a single photon which has leaked out of the cavity confirming the field mode. What is the field state inferred after such a detection? What is the field mode mean photon number inferred immediately after this leaking photon has been detected? Interpret this surprising result.

(3) Show that the ground state when a cavity field interacting with a two level atom is far detuned from resonance by Δ is Stark shifted in energy by interaction by an amount

$$\delta E = \frac{|\langle 1|V|2\rangle|^2}{\Delta\hbar} \tag{13.191}$$

(4) Explain why a coherent state is a good mathematical representation of typical laser light. Describe briefly the basic fully quantum mechanical description of light matter interaction.
A two level atom interacts on resonance with a single mode light field. Suppose that the atom is initially excited and the field has n photons. The atom field interaction is described by the Jaynes–Cummings Hamiltonian

$$H = \hbar\lambda(\sigma_- a^\dagger + \sigma_+ a) \tag{13.192}$$

Explain the physical significance of this Hamiltonian and the meaning of all the symbols.
Prove that the joint state of the atom and field at some time t is given by

$$\Psi(t) = \cos\lambda_n t|e, n\rangle + i\sin\lambda_n t|g, n+1\rangle \tag{13.193}$$

Give the expression for λ_n.
Calculate the probability for the atom to be in the ground state and plot it as a function of time.
Now suppose that the field is initially in a coherent state of amplitude α instead of a number state. Using linearity of Schrödinger's equation and *without performing any calculation*

whatsoever write down the expression for the probability that after time t the atom is in the ground state.
Assume that $\alpha = 0$. After what time is this probability greater than a half?

(5) An atom has two energy levels, i and j, separated in energy by $\hbar\omega_{ij}$. It is subject to a small external time dependent monochromatic electromagnetic perturbation for a time T, oscillating at the frequency ω. You may assume that the perturbation has matrix elements $V_{ji} = V_{ij}^*$ between these states. Show that if the atom is initially in the state i, the probability of a transition to the state j is approximately

$$P_{ij} = 4|V_{ij}|^2 \frac{\sin^2((\omega_{ij} - \omega)T/2)}{(\hbar(\omega_{ij} - \omega))^2} \tag{13.194}$$

(i) Argue that the probability of the transition back from j to i is the same as that for the transition from i to j.
(ii) Show that, within the formalism employed, the transition rate grows linearly with time.
(iii) Why is the Einstein B coefficient independent of time (only a qualitative explanation required)?

(6) Suppose that we have the following two level atom and single filed mode interaction Hamiltonian:

$$H = (\sigma_-(a^\dagger)^4 + \sigma_+ a^4) \tag{13.195}$$

Compute the dynamics of the initial state $|e, 0\rangle$.

13.5.2 *Solutions 5*

(1) Here we have to compute $\langle n|D(\alpha)|0\rangle$. Therefore we need to be able to evaluate

$$\langle n|e^{\alpha a^\dagger}|0\rangle \tag{13.196}$$

To do so, you can use the fact that $e^x = \sum_n x^n/n!$, and then apply the creation operator algebra. The rest of the question can be answered straightforwardly from the relevant chapter in this book.

(2) If $|\Psi\rangle = (|0\rangle + |10\rangle)/\sqrt{2}$, then

$$\langle\Psi|a^\dagger a|\Psi\rangle = \frac{1}{2}(0 + 10) = 5 \tag{13.197}$$

If we detect a photon, that is like applying annihilation operator to the above state. We obtain

$$a(|0\rangle + |10\rangle)/\sqrt{2} = \sqrt{5}|9\rangle \tag{13.198}$$

This state is not normalized; we have to divide it by $\sqrt{5}$ to normalize it. Therefore the number state with nine photons is obtained at the end. The average (in fact it is exact) number of photons is nine. So, detection has increased the average number!

(3) The relevant states in this case are the ones that will "(Rabi) flop into each other":

$$|1\rangle = |g\rangle|n\rangle \tag{13.199}$$

$$|2\rangle = |e\rangle|n-1\rangle \tag{13.200}$$

with the corresponding energies $E_1 = E_g + n\hbar\omega$ and $E_2 = E_e + (n-1)\hbar\omega$. The Hamiltonian is given by

$$H = \begin{pmatrix} E_1 & V_{12} \\ V_{21} & E_2 \end{pmatrix} \tag{13.201}$$

We obtain energies from

$$H\Psi_\pm = E_\pm \Psi_\pm \tag{13.202}$$

The solution to this eigenvalue equation is

$$E_\pm = \frac{1}{2}(E_1 + E_2) \pm \frac{1}{2}[(E_1 - E_2)^2 + 4|V_{12}|^2]^{1/2} \tag{13.203}$$

Large $E_1 - E_2$ leads to

$$E_\pm = \frac{1}{2}(E_1 + E_2) \pm \frac{\hbar}{2}\Delta\frac{1}{2}[1 + 2|V_{12}|^2/\Delta^2]^{1/2} \tag{13.204}$$

$$\sim \frac{1}{2}[(E_1 + E_2) \pm \hbar\Delta \pm 2|V_{12}|^2/\hbar\Delta] \tag{13.205}$$

The last term is the so-called Stark shift.

(4) A coherent state is the minimum uncertainty state in position and momentum of a harmonic oscillator.
This is a natural Hamiltonian from the physical perspective as it says that when a photon is lost in the field it is absorbed by the atom and vice versa.
Going to the interaction picture the Schrödinger equation reduces to

$$\hbar\lambda(\sigma_- a^\dagger + \sigma_+ a)|\psi\rangle = i\hbar\frac{\partial}{\partial t}|\psi\rangle \tag{13.206}$$

We assume that there are n photons in the field. Then due to energy conservation only the following superposition is possible:

$$|\psi\rangle = c_1|e, n\rangle + c_2|g, n+1\rangle \tag{13.207}$$

The Schrödinger equation becomes

$$\lambda(c_1\sqrt{n+1}|g, n+1\rangle + c_2\sqrt{n+1}|e, n\rangle) = i(\dot{c}_1|e, n\rangle + \dot{c}_2|g, n+1\rangle \tag{13.208}$$

Multiplying $\langle g|$ and $\langle e|$, we obtain

$$\lambda\sqrt{n+1}c_1 = i\dot{c}_2 \tag{13.209}$$

$$\lambda\sqrt{n+1}c_2 = i\dot{c}_1 \tag{13.210}$$

By taking the derivative of the second equation and substituting it into the first,

$$\ddot{c}_1 + (\lambda\sqrt{n+1})^2 c_1 = 0 \tag{13.211}$$

The solution is

$$c_1(t) = A\sin(\lambda\sqrt{n+1}t) + B\cos(\lambda\sqrt{n+1}t) \tag{13.212}$$

Therefore $\lambda_n = \lambda\sqrt{n+1}$.
But at $t = 0$, $c_1(0) = 0$, so that

$$c_1(t) = \sin(\lambda\sqrt{n+1}t) \tag{13.213}$$

The probability is therefore

$$p_1(t) = |c_1(t)|^2 = \sin^2(\lambda\sqrt{n+1}t) \tag{13.214}$$

If the field is in the coherent state

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{13.215}$$

then the amplitude for the ground state at time t is

$$c_1(t) = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}\sin(\lambda\sqrt{n+1}t) \tag{13.216}$$

Thus the probability is

$$p_1(t) = e^{-|\alpha|^2}\left|\sum_n \frac{\alpha^n}{\sqrt{n!}}\sin(\lambda\sqrt{n+1}t)\right|^2 \tag{13.217}$$

If $\alpha = 0$, then

$$p_1(t) = |\sin(\lambda t)|^2 \tag{13.218}$$

and $p_1 = 1/2$ implies $\sin(\lambda t) = 1/\sqrt{2}$, hence

$$t = \pi/4\lambda \tag{13.219}$$

(5) This is bookwork. Here is just one way of deriving it. If we divide the total time interval T into n small time intervals, then the amplitude for the transition is given by

$$\langle j|(1 - \frac{i}{\hbar}\tilde{V}(t)dt)^n|i\rangle \tag{13.220}$$

which is, up to the first order, equal to

$$\langle j|(1 - \frac{i}{\hbar}\int \tilde{V}(t)dt)|i\rangle \tag{13.221}$$

This is in the interaction picture, so converting back we have that

$$\tilde{V}(t) = V(t)e^{i\omega_{ij}t} \tag{13.222}$$

Therefore, the probability is given by the mod square of this

$$\frac{1}{\hbar^2}|\langle j|V\int e^{-i\omega t}e^{i\omega_{ij}}dt|i\rangle|^2 \tag{13.223}$$

(the positive frequency is omitted in the rotating wave approximation). Performing the integration leads to the required formula

$$P_{ij} = 4|V_{ij}|^2\frac{\sin^2((\omega_{ij}-\omega)T/2)}{(\hbar(\omega_{ij}-\omega))^2} \tag{13.224}$$

where $V = \langle j|V|i\rangle$ is the transition matrix element.

(i) The transition from j to i has $|V_{ji}|^2$, but $V_{ji} = V_{ij}^*$ and so $|V_{ji}| = |V_{ij}|$, and this proves the equality of the rates.

(ii) For a short time rate we have that (as can be shown by Taylor's expansion)

$$dP/dt = |V_{ij}|^2 T \tag{13.225}$$

as required.

(iii) To obtain B, we have to average over a continuum of states (of the system or the driving field — it doesn't matter which). Once we sum up over all the states that contribute to the transition, we obtain an expression that is independent of time. This is because the integral of the above sinc function is proportional to T, so that the resulting derivative is independent of it.

(6) We know how to solve the Jaynes–Cummings model. This interaction Hamiltonian is the same apart from the fact that it involves fourth power of the creation and annihilation operators. This means population oscillations at the rate $\cos \Omega_4 t$ where $\Omega_4 = \lambda\sqrt{(n+1)n(n-1)(n-2)}$ between the state $|e, n\rangle$ and $|g, n+4\rangle$.