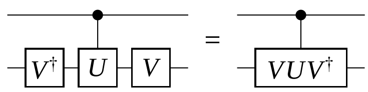
1 Exercises: quantum computation

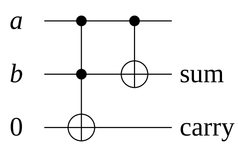
These exercises are closely based on those in the forthcoming book by Jones and Jaksch, and the numbering of sections follows the order in that book. In a few cases the exercises have been slightly rewritten so they stand alone. *These exercises are intended for self study and worked answers are provided on the course website; you are very strongly advised to make a serious attempt at all these exercises rather than just looking up the answer!* The exercises in section 9, with the exception of 9.1, are beyond the scope of the Oxford course.

7. Principles of quantum computing

1. Show that for reversible classical computing CLONE and SWAP gates can be built out of networks of controlled-NOT gates. Can these networks also be used for quantum computing?
2. Show how to build NOT and controlled-NOT gates from Toffoli gates. Design a reversible OR gate using only Toffoli gates and NOT gates.
3. The Fredkin gate is a three bit gate which swaps its two target bits if the single control bit is set to 1. Show how a Fredkin gate can be used to implement reversible NOT and AND gates.
4. Use your network for a SWAP gate to show how a Fredkin gate can be built from three Toffoli gates. Is it possible to build a Toffoli gate using only Fredkin gates?
5. Explain why the network identity

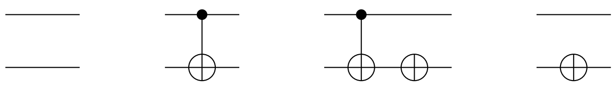
   works (note that the apparent reversal in the order of the operators simply reflects the different ordering conventions for operators and networks, and that $VV^\dagger = \mathbb{1}$). Use this identity to construct a Fredkin gate using only a single Toffoli gate and two controlled-NOT gates.
6. Consider the reversible half adder network

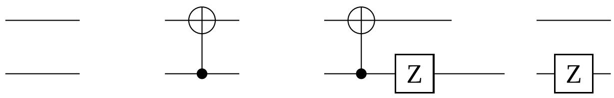
   Explain how this network works. Why isn't it necessary to preserve the second input?

8. Elementary quantum algorithms

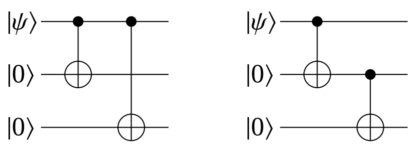
1. Given an oracle U_f implementing a function from one bit to one bit, design a classical circuit to directly determine the parity of f in two queries using only two bits (that is you may not store results "offline" for later comparisons and you may not use additional ancilla bits).
2. Consider the explicit circuits corresponding to the four functions f_{ij} in Deutsch's algorithm.



   Find an alternative circuit for f_{10} which applies single qubit gates to the upper qubit rather than the lower qubit.
3. Prove that applying Hadamard gates to both qubits before and after the circuits shown above gives the circuits shown below after simplification.



4. Calculate an explicit matrix form for the Grover amplitude amplification operator in the case $n = 2$, and hence show that Grover's quantum search will reveal a single satisfying input in a single query (you may neglect the ancilla qubit and use an explicit phase shift form for the action of the controlled gate on the two main qubits).
5. What happens in Grover's algorithm if the function has *two* satisfying inputs? What about three?
6. Show that the two encoding networks for quantum error correction



   will act as desired, and write down corresponding decoding networks.
7. Consider the three qubit spin flip error correcting network. By working through the network, find kets describing the state of the device immediately before the ancilla qubits are measured for an arbitrary logical input with each of the three single qubit errors or no error. Show that these states can be written as product states of the logical qubit and the ancilla qubits, and hence show that measuring the ancillas has no effect on the logical qubit.
8. Give explicit forms for the error correcting steps in the three qubit spin flip error correcting network (that is, what correction operators should be applied for each correction

outcome). Show how this process can be replaced by quantum control (replacing measurements and optional gates by conditional logic gates); state two disadvantages of this latter approach.

9. What happens to a classical bit protected with a three bit code if two bit flip errors occur? What happens in the quantum case?
10. Consider a single qubit which starts in the pure state $|\psi\rangle = \alpha|0\rangle+\beta|1\rangle$ and then undergoes one of two sorts of decoherence: (a) a rotation around the z-axis through an angle of either ϕ or $-\phi$, chosen at random; (b) experiencing a Z gate with probability p or being left alone with probability $1 - p$. Show that the density matrix description of these two cases is fundamentally the same, and determine the relationship between p and ϕ.

9. More advanced quantum algorithms

1. Consider a Deutsch–Jozsa problem with $n = 2$: how many possible functions are there, and how many are constant and how many are balanced? Assuming that an unknown function is known to be either constant or balanced with 50% probability, calculate the minimum, maximum, and average number of queries required to determine which sort of function it is on a classical computer. What about a quantum computer?

10. Trapped atoms and ions

1. Write down the potential energy function for a group of ions (each of mass M and charge $+e$) in a linear Paul trap, with strong radial and weak axial confinement. You may assume that the trap potentials are harmonic.
2. What effect does the motion of an ion have on its spectral lines if the ions is travelling in free space? What changes if the ion is confined in a harmonic trap?
3. Draw a quantum network based on the the collisional phase gate

$$U_\phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-i\phi} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

to implement a controlled-NOT gate with the first qubit as control and the second qubit as target.

4. Show that the collisional phase gate U_π can be written as $|0\rangle\langle 0| \otimes \mathrm{Z} + |1\rangle\langle 1| \otimes \mathbb{1}$. Hence show that the "massive entanglement" state of a system of two atoms can be written as

$$(|0\rangle \mathrm{Z} + |1\rangle)(|0\rangle + |1\rangle)$$

neglecting normalisation. How would you write the equivalent state for three atoms? Multiply this out to show that you agree with the result

$$|000\rangle \xrightarrow{\mathrm{H}^{(3)}} \xrightarrow{U_\pi} \frac{|+\rangle|0\rangle|-\rangle - |-\rangle|1\rangle|+\rangle}{\sqrt{2}}$$

11. Nuclear magnetic resonance

1. Estimate the strength of the magnetic field gradient required to make two ^{1}H nuclei in a molecule (assume a separation of about 1 Å) have Larmor frequencies differing by about 100 Hz. Would it be possible to obtain a gradient of this size?
2. Show that a Heisenberg coupling in a two spin system can be approximated by an Ising coupling as long as $|\omega_{12}| \ll |\omega_1 - \omega_2|$.
3. Draw an explicit network of gates to implement a controlled-NOT gate in a two spin system, using only standard single qubit gates and the gate $U(t)$ which corresponds to free evolution under the system's Hamiltonian for a time t. Draw an implementation of a NOT gate that takes the same length of time.
4. Consider a system of three coupled spins. Write down the Hamiltonian and then design a spin echo sequence such that the average Hamiltonian is reduced to a single coupling term between the second and third spins.
5. Return to a one spin system, and design a spin echo style sequence which will reduce the spin's apparent Larmor frequency to one half of its true value. Is it possible to change the sign of a spin's apparent Larmor frequency? What are the limits on the possible range of scalings? Can coupling strengths be rescaled in the same way?