# 1 Answers: quantum information

## 1. Quantum bits and quantum gates

1. This follows by direct matrix calculation

$$
\begin{aligned}
|\psi\rangle\langle\psi| &= \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{i\phi} \end{pmatrix} \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2)e^{-i\phi} \end{pmatrix} \\
&= \begin{pmatrix} \cos^2(\theta/2) & \sin(\theta/2)\cos(\theta/2)e^{-i\phi} \\ \sin(\theta/2)\cos(\theta/2)e^{i\phi} & \sin^2(\theta/2) \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} \cos\theta + 1 & \sin\theta(\cos\phi - i\sin\phi) \\ \sin\theta(\cos\phi + i\sin\phi) & -\cos\theta + 1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\cos\theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{\sin\theta\cos\phi}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{\sin\theta\sin\phi}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
&= \tfrac{1}{2}\left(\sigma_0 + s_x\,\sigma_x + s_y\,\sigma_y + s_z\,\sigma_z\right)
\end{aligned}
$$

with $s_x = \sin\theta\cos\phi$, $s_y = \sin\theta\sin\phi$, and $s_z = \cos\theta$. Thus

$$\mathbf{s.s} = \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta = 1$$

As $\mathbf{s}$ is a unit vector, it connects the origin with a point on the unit sphere.

2. A mixed state has the form $\rho = \sum_n P_n|\psi_n\rangle\langle\psi_n|$ where $P_n \geq 0$ and $\sum_n P_n = 1$. Each contributing density matrix can be described by a Bloch vector and so the mixed state can also be represented by a vector $\mathbf{s} = \sum_n P_n\mathbf{s}_n$. As all the $\mathbf{s}_n$ are of unit length, the weighted sum has a length of at most 1 (which only occurs when all the $P_n$ except one are zero). Thus the mixed state vector lies *inside* the unit sphere (the Bloch sphere). The point $\frac{1}{2}\sigma_0$ corresponds to the centre of the Bloch sphere. This is the *maximally mixed state*.

3. By direct multiplication $\sigma_x^2 = \sigma_0$ and similarly for $\sigma_y$ and $\sigma_z$. Now

$$
\begin{aligned}
\exp(-i\theta\,\sigma_\alpha/2) &= \sigma_0 + \left(\frac{-i\theta/2}{1}\right)\sigma_\alpha + \left(\frac{(-i\theta/2)^2}{2}\right)\sigma^2\alpha + \left(\frac{(-i\theta/2)^3}{3!}\right)\sigma^3\alpha + \ldots \\
&= \sigma_0 - i\left(\frac{\theta/2}{1}\right)\sigma_\alpha - \left(\frac{(\theta/2)^2}{2}\right)\sigma_0 - i\left(\frac{(-\theta/2)^3}{3!}\right)\sigma\alpha + \ldots \\
&= \sigma_0\left(1 - \frac{(\theta/2)^2}{2} + \ldots\right) - i\sigma_\alpha\left(\theta/2 - \frac{(\theta/2)^3}{3!} + \ldots\right) \\
&= \sigma_0\cos(\theta/2) - i\sigma_\alpha\sin(\theta/2)
\end{aligned}
$$

4. Using methods from above we note that the propagator for $90_y$ is $(\sigma_0 - i\sigma_y)/\sqrt{2}$, while

that for $180_x$ is $-i\sigma_x$. Then use brute force multiplication (note the order!)

$$\frac{-\mathrm{i}}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix} = \frac{-\mathrm{i}}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

which is the Hadamard gate (up to an irrelevant global phase). The other three are done in the same way.

5. $\mathrm{H}\sigma_x\mathrm{H} = \mathrm{H}\,(\mathrm{H}\sigma_z\mathrm{H})\,\mathrm{H} = (\mathrm{HH})\,\sigma_z\,(\mathrm{HH}) = \mathbb{1}\sigma_z\mathbb{1} = \sigma_z$.
6. Reversing the definitions of $|+\rangle$ and $|-\rangle$ gives $|0\rangle = (|+\rangle + |-\rangle)/\sqrt{2}$ and $|1\rangle = (|+\rangle - |-\rangle)/\sqrt{2}$, so

$$|\psi\rangle = \left(\frac{\alpha+\beta}{\sqrt{2}}\right)|+\rangle + \left(\frac{\alpha-\beta}{\sqrt{2}}\right)|-\rangle$$

and making an X-measurement returns $|+\rangle$ or $|-\rangle$ with probabilities

$$P_+ = \left(\frac{\alpha+\beta}{\sqrt{2}}\right)^2 \qquad P_- = \left(\frac{\alpha-\beta}{\sqrt{2}}\right)^2.$$

Alternatively we have

$$\mathrm{H}|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\alpha+\beta\\ \alpha-\beta\end{pmatrix}$$

and making a Z-measurement gives $|0\rangle$ or $|1\rangle$ with probabilities as above; applying a final Hadamard gate simply converts these to $|+\rangle$ and $|-\rangle$ with the same probabilities.
7. The first Hadamard gate rotates the two eigenstates of the X-measurement onto the two eigenstates of a Z-measurement. Any single qubit measurement will have two eigenstates which lie at diametrically opposed points on the Bloch sphere and which can be rotated onto $|0\rangle$ and $|1\rangle$ in the same way. As rotation gates are unitary the will also work with superposition states. Equivalently we can think of unitary gates as rotating operators rather than states: surrounding a Z-measurement gate with a pair of Hadamard gates is equivalent to rotating it into a Z-measurement.

## 2. An atom in a laser field

1. In systems of this kind the single outer electron can be thought of as moving in a central field, although the form of this field will be much more complex than the simple Coulomb field found in hydrogen, and so the wavefunction will still be separable into radial and angular parts. As the selection rules only rely on certain angular integrals being zero they will be unaffected by the changes to the radial parts.
2. Inserting the numbers gives $1/\Gamma \approx 31$ ns. To achieve Rabi flopping we need $V \gg \Gamma$, and using $E = \hbar V/a_0 e$ gives $E \gg 400$ V for the *rotating* field; double this for the oscillating field. In reality the lifetime is about 7 ns, and the field must be large compared with 5000 V/m.
3. Sudden jumps are only effective when $V \gtrsim \omega_0$ or $E \gtrsim 2\pi\hbar c/ea_0\lambda$, and putting the numbers in gives $E \gtrsim 6\times10^{10}$ V/m, which is too large to generate as a static field (breakdown will occur). Even if you could produce the field it would cause many other transitions as well.

4. The Abbe limit is $\lambda/2 \approx 200\,\mathrm{nm}$ (realistic systems are often around an order of magnitude worse than this).
5. For the excited state population, compare the energy gap $E = hc/\lambda \approx 5\times10^{-19}\,\mathrm{J}$ with $k_BT \approx 4.1\times10^{-21}\,\mathrm{J}$ at 300 K; clearly the excited state population will be negligible.
6. To get a peak electric field strength of 800 V/m in a spot of diameter 200 nm requires a power of 13 pW; the surprisingly low power requirement largely reflects the tiny size of the laser spot. This calculations is fairly unrealistic, both as to the field strength required and the spot size achievable: using more realistic numbers of 10000 V/m and a diameter of $10\,\mu\mathrm{m}$ gives a power of $5\,\mu\mathrm{W}$. Significantly larger powers are used in real quantum computers as this enables Raman transitions to be used far from resonance.

## 3. Spins in magnetic fields

1. This can be worked out by brute force but it is simpler just to rescale the field. If a $90^\circ$ rotation lasts $6\,\mu\mathrm{s}$ then a $360^\circ$ rotation lasts $24\,\mu\mathrm{s}$, and the rotation rate is $10^6/24\,\mathrm{Hz}$. Divide this by 500 MHz and multiply by 12 T to get 1 mT or 10 Gauss. But this is the strength of the *rotating* field, and we want the oscillating field, so double this to get 2 mT or 20 Gauss.
2. Using $E = h\nu$ with $\nu = 500\,\mathrm{MHz}$ gives an energy gap of $3.313\times10^{-25}\,\mathrm{J}$ or $2\,\mu\mathrm{eV}$. Compare this with $k_BT$ at 300 K which is $4.142\times10^{-21}\,\mathrm{J}$, so the two states will be very nearly equally occupied. A Boltzmann calculation gives fractional populations of 0.50002 and 0.49998, with an excess fractional population of $4\times10^{-5}$.
3. In a sample of 0.2 ml of water there are about 0.2/18 moles of water, which is $6.7\times10^{21}$ molecules, but each molecule has two hydrogen atoms, giving $1.34\times10^{22}$ nuclei. Thus the excess population is $5.35\times10^{17}$ spins. For the last bit, solve the Boltzmann equation to discover that 99% population in the lower state requires $h\nu/k_BT = 4.595$ or a temperature of 5 mK.
4. The answer depends on what is meant by "tolerate", but suppose that we insist that the inhomogeneous broadening can be no more than 50% as large as the homogeneous broadening, that is 0.5 Hz. To achieve this at a frequency of 0.5 GHz requires a field homogeneous to one part in $10^9$. Reaching this limit is difficult and expensive, but possible over small regions of space.
5. Begin by finding the propagators for the underlying gates

$$\phi_z = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi/2} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\phi/2} \end{pmatrix} \qquad 180_x = -\mathrm{i}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad 180_y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and the first result is shown by direct multiplication

$$\begin{aligned} \phi_z\,180_x\,\phi_z &= -\mathrm{i}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi/2} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\phi/2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi/2} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\phi/2} \end{pmatrix} \\ &= -\mathrm{i}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi/2} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\phi/2} \end{pmatrix}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\phi/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\phi/2} \end{pmatrix} = -\mathrm{i}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

which is identical to $180_x$. The propagator for the second spin echo sequence is just the square of the above, and $(-\mathrm{i}\sigma_x)^2 = -\mathbb{1}$, which is the identity up to a global phase.

The third and fourth sequences give the same result. Direct multiplication shows that changing the phase of both 180 pulses has no effect, but changing just one of them gives $-\mathrm{i}\sigma_z$, which is a $180_z$ rotation. In general using 180 pulses separated by a phase angle is equivalent to performing a $z$-rotation through twice that angle.

## 4. Photon techniques

1. We have the general form for a wave plate

$$U(\theta,\phi) = \begin{pmatrix} \cos^2\theta + \sin^2\theta \mathrm{e}^{\mathrm{i}\phi} & \cos\theta\sin\theta(1-\mathrm{e}^{\mathrm{i}\phi}) \\ \cos\theta\sin\theta(1-\mathrm{e}^{\mathrm{i}\phi}) & \cos^2\theta\mathrm{e}^{\mathrm{i}\phi} + \sin^2\theta \end{pmatrix}$$

and a half wave plate corresponds to $\phi = \pi$ so

$$U(\theta,\pi) = \begin{pmatrix} \cos^2\theta - \sin^2\theta & 2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & -\cos^2\theta + \sin^2\theta \end{pmatrix} = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$$

Now choosing $2\theta = \pi/2$ gives $U(\pi/4,\pi) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which is a NOT gate, while choosing $2\theta = \pi/4$ gives

$$U(\pi/8,\pi) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

which is a Hadamard gate.

2. As the number states are all orthonormal we have

$$\langle\alpha|\alpha\rangle = \mathrm{e}^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{(\alpha^*\alpha)^n}{n!} = \mathrm{e}^{-|\alpha|^2}\mathrm{e}^{+|\alpha|^2} = 1$$

and $P(n) = \mathrm{e}^{-|\alpha|^2}|\alpha|^{2n}/n!$. Hence $P(0) = \mathrm{e}^{-|\alpha|^2}$, $P(1) = |\alpha|^2 P(0)$ and $P(n>0) = 1 - P(0)$. Thus if a laser pulse contains at least one photon then the probability that it contains exactly one photon is

$$P(n=1|n>0) = \frac{|\alpha|^2\mathrm{e}^{-|\alpha|^2}}{1-\mathrm{e}^{-|\alpha|^2}}$$

and for $\alpha = \sqrt{0.1}$ we get $P(0) = 0.9048$, $P(1) = 0.0905$, and $P(n=1|n>0) = 0.9508$.

## 5. Two qubits and beyond

1. Start by writing the controlled-NOT gate as $|0\rangle\langle 0|\otimes\mathbb{1} + |1\rangle\langle 1|\otimes\mathrm{X}$. Then note that $\mathrm{X} = \mathrm{HZH}$, and that $\mathbb{1}$ can be written as $\mathrm{H}\mathbb{1}\mathrm{H}$. Finally factor out the common Hadamard gate to leave $(\mathbb{1}\otimes\mathrm{H})\cdot(|0\rangle\langle 0|\otimes\mathbb{1} + |1\rangle\langle 1|\otimes\mathrm{Z})\cdot(\mathbb{1}\otimes\mathrm{H})$ as desired.
2. Start from a general eigenstate $|a\rangle|b\rangle$ and follow through the sequence to three controlled-NOT gates in turn using $a\oplus a\oplus b = 0\oplus b = b$ to simplify terms:

$$|a\rangle|b\rangle \rightarrow |a\rangle|a\oplus b\rangle \rightarrow |a\oplus(a\oplus b)\rangle|a\oplus b\rangle = |b\rangle|a\oplus b\rangle \rightarrow |b\rangle|(a\oplus b)\oplus b\rangle = |b\rangle|a\rangle. \quad (1.1)$$

The final part follows immediately from the linearity of unitary operations. This is normally thought of as swapping amplitudes between the two qubits, but it is often better to think of this process as swapping the labels identifying the two qubits.

3. The matrix form of the SWAP gate
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.2}$$
can be obtained either by matrix multiplication or by noting directly that it swaps $|01\rangle$ and $|10\rangle$ and leaves $|00\rangle$ and $|11\rangle$ alone. Explicit calculation shows that the SWAP gate leaves three Bell states entirely unaffected, except that the singlet state $|\psi^-\rangle$ picks up a global phase factor of $-1$. The result is not surprising as the Bell states are symmetric or antisymmetric states of the two qubits as a whole, not of the individual qubits, and so should not be affected by swapping the labels of the qubits. The global phase of $-1$ for $|\psi^-\rangle$ occurs because this state is antisymmetric under the exchange of qubit labels.
4. The explicit forms are
$$|00\rangle = \frac{|\phi^+\rangle + |\phi^-\rangle}{\sqrt{2}} \qquad |11\rangle = \frac{|\phi^+\rangle - |\phi^-\rangle}{\sqrt{2}} \tag{1.3}$$
$$|01\rangle = \frac{|\psi^+\rangle + |\psi^-\rangle}{\sqrt{2}} \qquad |10\rangle = \frac{|\psi^+\rangle - |\psi^-\rangle}{\sqrt{2}} \tag{1.4}$$
Just as superpositions of separable states can be entangled, so superpositions of entangled states can be separable.
5. Direct calculation shows that starting from $|1\rangle|0\rangle$ gives $|\phi^-\rangle$. Similarly starting from $|0\rangle|1\rangle$ and $|1\rangle|1\rangle$ gives $|\psi^+\rangle$ and $|\psi^-\rangle$ respectively.
6. Simply apply the gates in the entangling network in reverse order to dis-entangle the states. This works because both controlled-NOT and H are self-inverse; in general you would have to use inverse gates as well as reversing the order.

## 6. Measurement and entanglement

1. For the case of two polarizers the transmitted fraction is $(\cos(\pi/4))^4 = 1/4$. This generalizes in the obvious way to $(\cos(\pi/2n))^{2n}$, and in the infinite limit the transmitted fraction goes to one. Note that there is no loss at the first polarizer as we are using pre-aligned light. We are, of course, assuming perfect polarizers throughout.
2. Ignoring global phases a Z gate will turn $\psi^-$ into $\psi^+$, while an X gate will turn it into $\phi^-$. To obtain $\phi^+$ either apply both X and Z gates in either order, or note that this combination is equivalent to Y.
3. Clearly it suffices to show that $\psi^-$ is unaffected by bilateral Hadamards and T gates ($\mathrm{H} \otimes \mathrm{H}$ and $\mathrm{T} \otimes \mathrm{T}$). These are easily shown by direct calculation.
4. Measurement in any basis can be achieved by applying some rotation to the qubit before and after the measurement, and if the measurement bases are the same for each qubit then the rotations must be the same for each qubit. But this is a bilateral rotation, and we have just shown that $|\psi^-\rangle$ is invariant under bilateral rotations. The final state after the measurement will be affected by the second bilateral rotation, but this does not affect the outcome of the measurements. This argument does not work for the other

four Bell states as they are not invariant under bilateral rotations: for example $|\phi^{\pm}\rangle$ are interconverted by bilateral T gates, while $|\phi^{-}\rangle$ and $|\psi^{+}\rangle$ are interconverted by bilateral Hadamard gates.

5. There are many ways to argue this, all of which are essentially equivalent. After Alice applies her Hadamard her qubit will be in the state $(|0\rangle + |1\rangle)/\sqrt{2}$; she then makes her measurement and the state decoheres to the mixed state $(|0\rangle\langle 0| + |1\rangle\langle 1|)/2$. This is all local to Alice, so Bob's qubit is still in the state $|0\rangle$ and we can describe the whole system by the direct product

$$\tfrac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes |0\rangle\langle 0| = \tfrac{1}{2}(|00\rangle\langle 00| + |10\rangle\langle 10|). \tag{1.5}$$

Finally we consider the effect of Alice talking to Bob: if she got $|0\rangle$ then Bob does nothing and the state remains $|00\rangle\langle 00|$; if she got $|1\rangle$ then Bob applies a NOT gate to his qubit, converting $|10\rangle\langle 10|$ into $|11\rangle\langle 11|$. Thus the final state is $\frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$. This superficially looks like $|\phi^{+}\rangle\langle\phi^{+}|$, but writing it out in matrix form

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \tag{1.6}$$

we see that it is not the same. Indeed it is obviously not entangled as the form $\frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$ is a mixture of $|00\rangle$ and $|11\rangle$, that is a mixture of two separable states.

State fidelities are easy to obtain by direct multiplication: for $|\phi^{+}\rangle$ we get

$$\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{1.7}$$

which is $\frac{1}{2}$, and the same result is found for $|\phi^{-}\rangle$. The fidelity with $|\psi^{\pm}\rangle$ is found to be zero. This suggests that we can write the state as $\frac{1}{2}(|\phi^{+}\rangle\langle\phi^{+}| + |\phi^{-}\rangle\langle\phi^{-}|)$, and this is indeed the case. Thus a mixture of entangled states need not be entangled.