Non-linear Optics II
(Modulators & Harmonic Generation)

P.E.G. Baird
MT2012

Electro-optic modulation of light

An electro-optic crystal is essentially a variable phase plate and as such can be used either as an amplitude (intensity) modulator or as a phase modulator. One arrangement for this was shown in the last lecture. Of course the field may be applied in various directions and some examples are shown below.

We shall consider in a little detail here the longitudinal case.

The input beam is specified by

$$E_{x'} = E_{y'} = A; \ \ E_y = 0 \tag{1}$$

i.e. linearly polarised along x. The output is thus affected by the phase difference between the x' and y' directions which is induced by the electric field along z. The output field is thus,

$$E_{x'}(\ell) = A; \ \ E_{y'}(\ell) = A\exp(-i\phi) \tag{2}$$

The resultant complex field along y is then,

$$E_y(\ell) = \frac{A}{\sqrt{2}}\left(\exp(-i\phi) - 1\right) \tag{3}$$

and the transmission ratio is

$$I_{out} = I_{in}\sin^2(\phi/2) \tag{4}$$

where

$$\phi = \frac{\pi V}{V_{\lambda/2}} \tag{5}$$

The quarter-wave plate allows modulation about the linear part of the transmission curve for fields close to zero.

A similar intensity modulator can be constructed using a Mach-Zehnder arrangement with the crystal in one of the two paths. Electro-optic modulation effectively scans the path difference of the two beam interferometer and therefore scans from say an intensity maximum to a minimum. Biasing can also be achieved using an additional $\lambda/4$ plate as above.
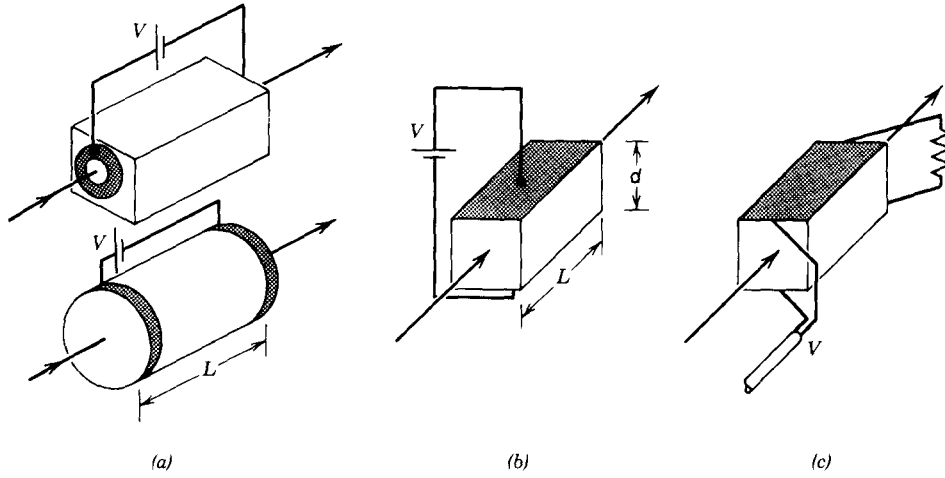
Figure 1: (a) a longitundial field, (b) a transverse field, and (c) a travelling-wave field.
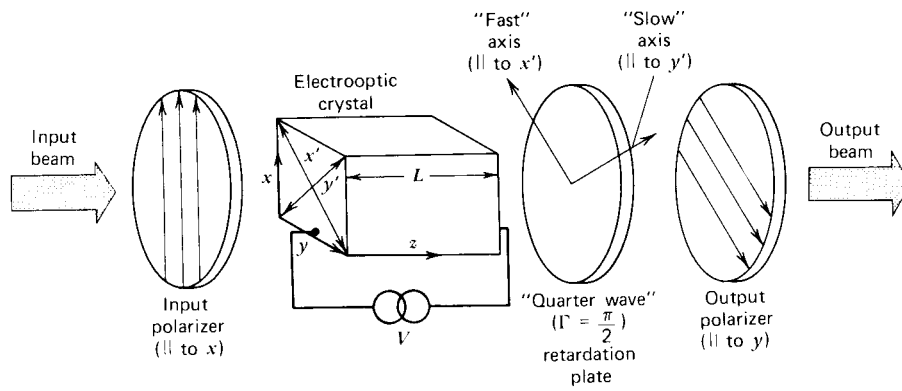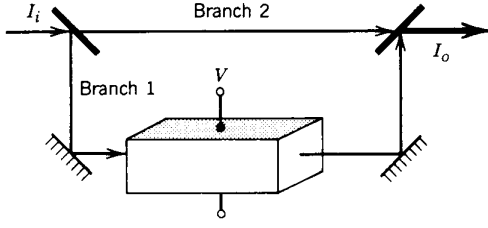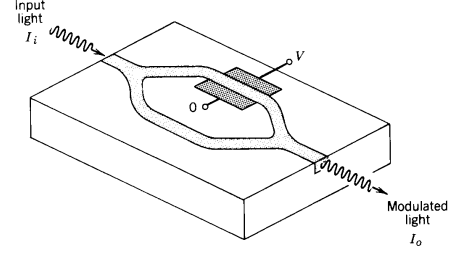

Figure 2: Longitundal modulator. The $\lambda/4$ plate provides a "bias" to 50% transmission

Principle of the Mach-Zehnder
Interferometer

Optical fibre version of a Mach-Zehnder

Phase modulators.

Using now only the input polariser along say x', a varying voltage to the crystal can lead to phase and frequency modulation as follows. Let the light field be specified by

$$E_{in} \;=\; A\cos\omega t \tag{6}$$

$$E_{out} \;=\; A\cos(\omega t - kx + \Delta\phi) = A\cos(\omega t - \frac{\omega}{c}(n_0 - \frac{n_0^3}{2}r_{63}E_m\sin\Omega t)\ell) \tag{7}$$

and the applied voltage by

$$V_m = V_0\sin\Omega t \tag{8}$$

Then, the phase of the light is modulated like $\Delta\phi = \delta\sin\Omega t$ where the modulation index is given by

$$\delta = \frac{1}{2}\frac{2\pi}{\lambda}n_0^3 r_{63}V_0 = \omega_0 n_0^3 r_{63}V_0/(2c) \tag{9}$$

Omitting the constant phase factor $\omega\ell n_0/c$, the output light field is therefore described by,

$$E_{out} = A\cos(\omega_0 t + \delta\sin\Omega t) \tag{10}$$

which can be written as a series of Bessel functions,
$$E_{out} = A[J_0(\delta)\cos\omega_0 t + J_1(\delta)\left\{\cos(\omega_0 + \Omega)t - \cos(\omega_0 - \Omega)t\right\}$$
$$+J_2(\delta)\left\{\cos(\omega_0 + 2\Omega)t + \cos(\omega_0 - 2\Omega)t\right\}..]$$

where the $J_n(\delta)$ represents a Bessel function of the n th order. The result is that the *spectrum* of the output light now contains sidebands shifted by harmonics of the modulator frequency either side of the central carrier (laser) frequency, i.e. $\omega \pm n\Omega$.

Electro-optic deflection of light

An optical beam can be dynamically deflected by electrical control of the refractive index of a prism. The angle of deflection produced by a prism with small apex angle α and refractive index n is $\theta \sim (n-1)\alpha$. A change in refractive index Δn caused by an electric field E corresponds to a change in the deflection angle of

$$\Delta\theta = \alpha\Delta n = -\frac{1}{2}\alpha r n^3 E = -\frac{1}{2}\alpha r n^3 \frac{V}{d} \tag{11}$$

Note here the factor of 1/2 because of the way the crystal is cut: voltage applied along the $z-$direction; light polarised along $x'-$direction (direction of dimension D) and y'-direction coincident with dimension L.
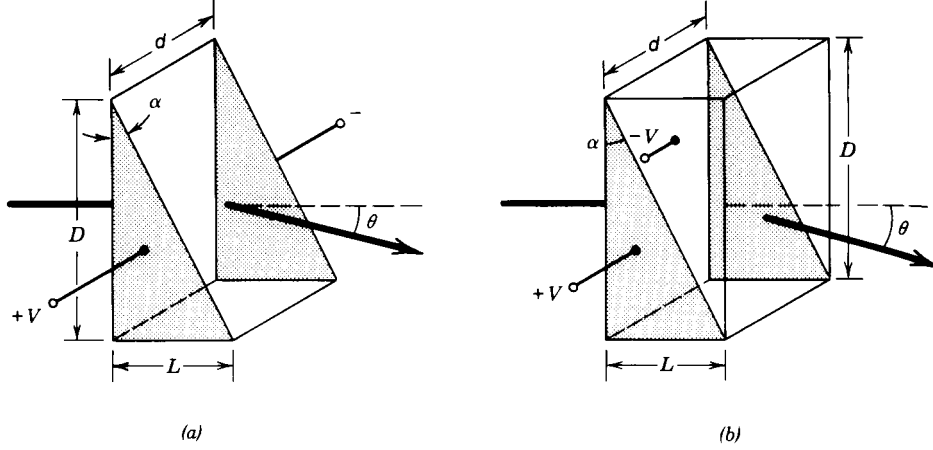
Figure 3: Electro-optic deflection of light.

The resolution of the scanner is of course determined by the angular diffraction $\delta\theta \sim \lambda_0/D$ where D is the diameter of the incident laser beam. To minimise this the beam should be as large as possible and fill the aperture of the prism. Unfortunately as d is increased so to must V to maintain the same electric field strength E on the crystal.

Acousto-optic deflection of light

We also mention here as an aside that modulation of light beams by sound waves is possible and that acousto-optic devices are commonly used to scan the frequency of a laser beam or to modulate its intensity. Acousto-optic effects come under the headings of: Bragg scattering, Raman-Nath or Debeye-Sear scattering and Brillouin scattering. As an illustration we show below the situation for an acousto-optic device operating in the Bragg regime *i.e.*,where the incident light interacts with broad sound wave, phase-fronts. The relevant Bragg condition is then,

$$\sin\theta = \frac{m\lambda}{2\lambda_s} \tag{12}$$

where λ_s refers to the sound wave and $m = 1$ for a sinusoidal sound wave disturbance. The sound wave moving in the upward direction with velocity v_s^+ Doppler shifts the light wave to frequency $v + f_s$; similarly the downward propagating sound wave down-shifts the light to frequency $v - f_s$. Equivalently we may say that an incident photon with wave-vector k_1 collides with a quantised particle of acoustic energy (a phonon) with wave-vector k_s to create a photon of wave-vector k_2 which satisfies momentum conservation through the relationship

$$k_2 = k_1 \pm k_s \tag{13}$$

Harmonic Generation

In lecture 1 (equation 4) we saw that for a certain class of crystals it was possible to have a $\chi^{(2)}$ effect and thus to induce a polarisation oscillating at the second harmonic frequency using two identical laser fields. In the equation below we take account there being two equivalent terms ($E(\omega_1)E(\omega_2) = E(\omega_2)E(\omega_1)$) by inserting the factor of 1/2, thus,

$$P^{2\omega} = \varepsilon_0 \frac{\chi^{SHG}}{2} E^\omega E^\omega \tag{14}$$
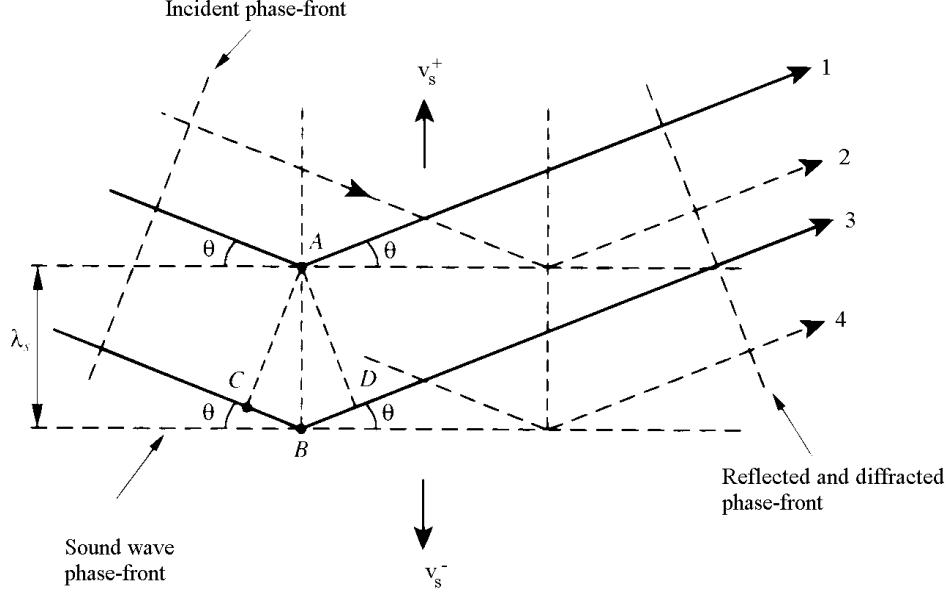
Figure 4: Bragg condition for diffraction from a sound wave.

The equation for the second harmonic wave is then

$$\frac{d^2 E^{2\omega}}{dz^2} + \left(\frac{2\omega \; n^{2\omega}}{c}\right)^2 E^{2\omega} = -\mu_0 (2\omega)^2 P^{2\omega} \tag{15}$$

while that for the fundamental wave is

$$\frac{d^2 E^{\omega}}{dz^2} + \left(\frac{\omega \; n^{\omega}}{c}\right)^2 E^{\omega} = -\mu_0 (\omega)^2 P^{\omega} \tag{16}$$

Solutions are of the form

$$E^{\omega} = A^{\omega} \exp[ik^{\omega}z]; \;\; E^{2\omega} = A^{2\omega} \exp[ik^{2\omega}z] \tag{17}$$

where $A_{\omega} = A_0(z) \exp(-i\omega t)$

Substituting solutions of this form into 15, and replacing $P^{2\omega}$ using 14 we find the second harmonic amplitude is given by,

$$\frac{dA^{2\omega}}{dz} = \left(\frac{i(2\omega \frac{1}{2}\chi^{SHG})(A^{\omega})^2}{2n^{2\omega}c}\right) \exp(i\Delta kz) \tag{18}$$

where the wave vector mismatch is

$$\Delta k = 2k^{\omega} - k^{2\omega} \tag{19}$$

The intensity (Poynting's vector[1]) is thus

---

[1] Maxwell's equations lead to the following continuity equation in the absence of any Joule heating term.

$$\nabla.\mathbf{S} + \frac{\partial \rho}{\partial t} = 0$$
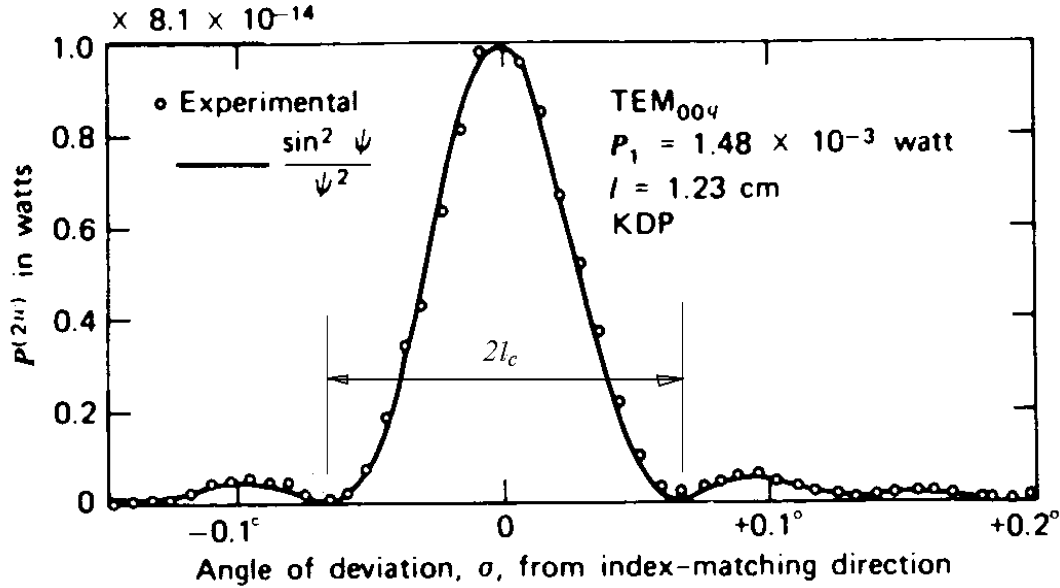
Or using the divergence theorem,

Figure 5: Effect of phase-matching in KDP. The graph shows the variation in SHG ouptut as a function of the phase-match angle.

$$I^{2\omega} = \frac{1}{2}n^{2\omega}\sqrt{\frac{\varepsilon_0}{\mu_0}}\left|A^{2\omega}\right|^2 \tag{20}$$

so that integrating 18 we have finally,

$$I^{2\omega} = \frac{(2\omega)^2(\frac{1}{2}\chi^{SHG})^2}{2n^{2\omega}(n^\omega)^2 c^3\varepsilon_0}(I^\omega)^2\left\{\frac{\sin(\Delta kz/2)}{\Delta kz/2}\right\}^2 z^2 \tag{21}$$

This is a maximum when $z = 2\pi/\Delta k = \ell_c$ which introduces the coherence length ℓ_c

References

[1] *Laser Phase and Frequency Stabilization Using an Optical Resonator* R.W.P Drever et al *Applied Physics* **B31**, *97 1983*

[2] *Laser frequency stabilisation by polarisation spectroscopy of a reflecting reference cavity.* T.W. Hänsch and B. Couillaud *Optics communications* **35**, 441, 1980

[3] *Fundamentals of Photonics* B.E.A. Saleh and M.C. Teich (Wiley 1994) Chapters 18 & 20

[4] *Lasers and Electro-optics (Fundamental and Engineering)* (CUP 1994) Chapter 19

$$\int \mathbf{S}.d\mathbf{A} = -\frac{\partial}{\partial t}\int \rho dV$$

where the energy density ρ is given by $\frac{1}{2}\{\mathbf{E}.\mathbf{D}+\mathbf{H}.\mathbf{B}\}$ and Poynting's vector $\mathbf{S}$ is equal to $\frac{1}{2}\operatorname{Re}\{\mathbf{E}\times\mathbf{H}^*\}$.Thus the in or outward flow of energy over a surface is equal to the change in the energy density bounded by the surface. For a plane wave in vacuo E and H are related by the intrinsic impedance $Z_0 = \sqrt{\mu_0/\varepsilon_0}$. Note that the optical intensity is equal to the magnitude of the time-averaged Poynting vector, *i.e.*,$I = \langle S\rangle$

Appendix A: The missing steps

Start by taking the curl of the curl E equation and include the polarisation in two parts: linear and non-linear

$$\nabla \times (\nabla \times E) = \nabla(\nabla . E) - \nabla^2 E = \nabla \times \left(-\frac{\partial B}{\partial t} \right) \tag{22}$$

$$\nabla^2 E = \frac{\partial}{\partial t}(\nabla \times \mu_0 H) = \frac{\partial}{\partial t}\left(\mu_0 J + \mu_0 \frac{\partial D}{\partial t} \right) \tag{23}$$

$$= \mu_0 \frac{\partial^2}{\partial t^2}(\varepsilon_0 E + P_L + P_{NL}) = \mu_0 \varepsilon_0 \varepsilon_r \frac{\partial^2 E}{\partial t^2} + \mu_0 \frac{\partial^2 P_{NL}}{\partial t^2} \tag{24}$$

$$\nabla^2 E - \mu_0 \varepsilon_0 \varepsilon_r \frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P_{NL}}{\partial t^2} \tag{25}$$

Now make the following substitutions:

$$P_{NL} = \varepsilon_0 \frac{\chi}{2} E_\omega E_\omega; \quad E_\omega = A_\omega e^{ik_\omega z} \tag{26}$$

Take the case of plane waves (1D - case) propagating along z, then the 2ω equation (see equation 15) becomes:

$$\frac{d^2 E_{2\omega}}{dz^2} + \frac{(2\omega)^2 n^2}{c^2} E_{2\omega} = -\mu_0 (2\omega)^2 P_{2\omega} \tag{27}$$

$$-k_{2\omega}^2 A_{2\omega} + 2ik_{2\omega}\frac{dA_{2\omega}}{dz} + \left(\frac{n_{2\omega} 2\omega}{c} \right)^2 A_{2\omega} = -\mu_0 (2\omega)^2 \varepsilon_0 \frac{\chi}{2} A_\omega^2 \tag{28}$$

The first and third terms on the LHS of the equation are equal and opposite so that we get:

$$\frac{i(2\omega)^2 A_\omega^2 \chi \exp(i\Delta k z)}{2c^2 2k_{2\omega}} = \frac{i\left(2\omega \frac{1}{2}\chi^{SHG} \right) A_\omega^2}{2n_{2\omega}c} \exp(i\Delta k z) = \frac{dA_{2\omega}}{dz} \tag{29}$$

where we have assumed a small variation of A with z and $\Delta k = 2k_\omega - k_{2\omega}$. Thus, the equation reduces as follows:

$$\Longrightarrow \int_0^z \exp(i\Delta k z)\, dz \Longrightarrow sinc\ \{\} \tag{30}$$

Finally, noting $I_{2\omega} \propto A_{2\omega}^2$ this leads directly to equations 21.

Appendix B: Use of Modulators to Stabilise and to Scan Lasers.

The Pound-Drever-Hall Technique

The experimental layout below illustrates the typical configuration for the Pound-Drever-Hall locking arrangement [1]. In this case the cavity follows the variations in the wavelength of the laser such that an integral number of half-wavelengths exist between the cavity mirrors. The role of the Pockels cell is to phase-modulate the laser output so that there are sidebands imposed on the laser's output. The frequency of phase-modulation and therefore the separation of the sidebands from the central carrier frequency is derived from the local oscillator (LO) The back-reflected beam from the cavity is detected on a fast photodiode and demodulated by the mixer to provide the error signal which then servo-controls a piezo-electric actuator which corrects the cavity length. Now let's see what information is contained in the back-reflected beam. We will write $E_{ref} = F E_{inc}$. There may be some phase shift n reflections but this can be taken into account by allowing the reflection coefficient F to be complex. Specifically in the absence of loses for a symmetric cavity

$$F = \frac{r \ (e^{i\phi} - 1)}{1 - r^2 e^{i\phi}} \tag{31}$$

where r is the reflection amplitude at each mirror and ϕ the phase shift that is picked up in a round trip within the cavity. We may relate the phase ϕ to the $\Delta\nu_{fsr} = c/2L$ as follows

$$\phi = \frac{\omega}{\Delta\nu_{fsr}} = 2\pi \times \frac{f}{\Delta\nu_{fsr}} \tag{32}$$

or the round trip optical path in the cavity in terms of the wavelength of the light

$$\phi = 2\pi \times \frac{2L}{\lambda} \tag{33}$$

The reflection coefficient is periodic in terms of length or laser frequency; the phase changes by 2π for a length change of $\lambda/2$. To calculate the field of the reflected beam when there are several different frequency components entering the cavity you need to multiply each by the reflection coefficient at the relevant frequency.
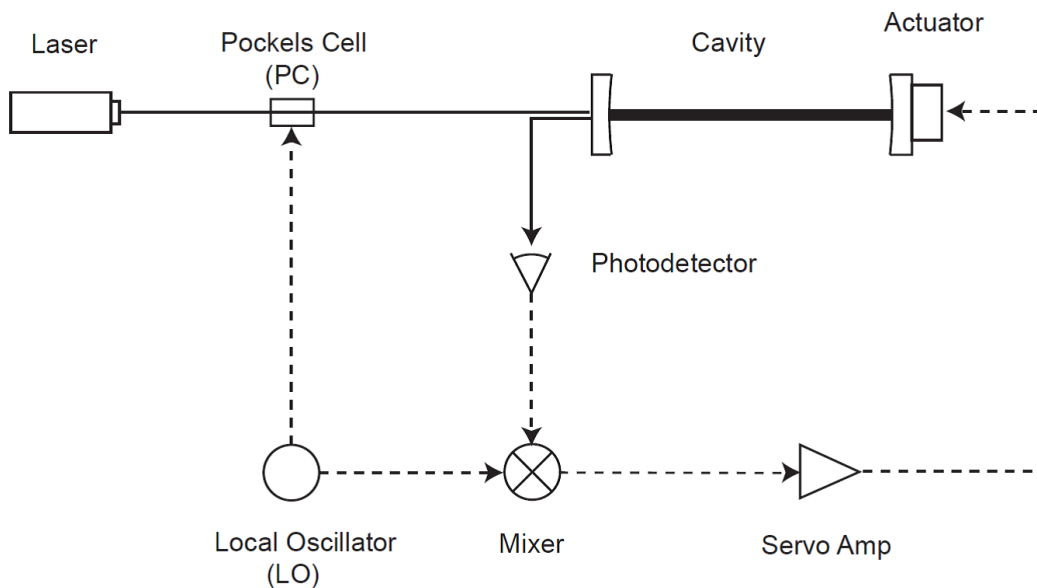

Figure 6: Typical arrangement for Pound-Drever-Hall Locking

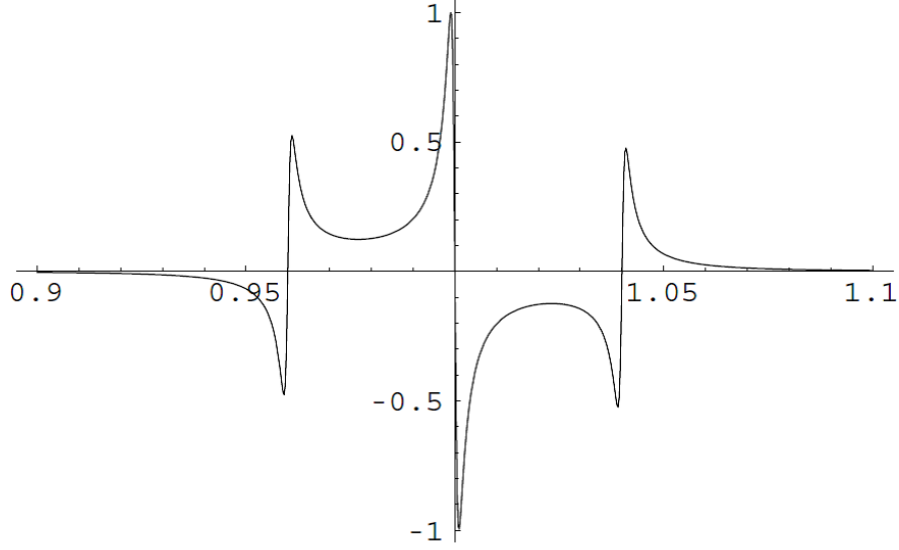Thus, considering only the nearest sidebands, we get

Figure 7: Plot of the error signal $\varepsilon/(2\sqrt{P_c P_s})$ against ϕ. The modulation is ˜4% of the FSR and the cavity finesse is ˜100

$$E_{ref} = E_0 \left[F(\omega) J_0(\beta) e^{i\omega t} + F(\omega + \Omega) J_1(\beta) e^{i(\omega + \Omega)t} - F(\omega - \Omega) J_1(\beta) e^{i(\omega - \Omega)t} \right] \qquad (34)$$

In terms of the power which is detected by the photodiode we have

$$
\begin{aligned}
|E_{ref}|^2 &= P_{ref} = P_c |F(\omega)|^2 + P_s \{|F(\omega + \Omega)|^2 + |F(\omega - \Omega)|^2\} & (35) \\
&\quad + 2\sqrt{P_c P_s} \operatorname{Re}\left[F(\omega) F(\omega + \Omega)^* - F(\omega)^* F(\omega - \Omega) \right] \cos \Omega t & (36) \\
&\quad + \operatorname{Im}\left[F(\omega) F(\omega + \Omega)^* - F(\omega)^* F(\omega - \Omega) \right] \sin \Omega t\} + (2\Omega - terms) & (37)
\end{aligned}
$$

The mixer which operates just like a phase-senstive detector pulls out the term proportional to $\sin \Omega t$ so that the error signal is

$$\varepsilon = 2\sqrt{P_c P_s} \operatorname{Im}\left[F(\omega) F(\omega + \Omega)^* - F(\omega)^* F(\omega - \Omega) \right] \qquad (38)$$

Note here that the slopes of the zero-crossings of the carrier and sideband components are opposite and one would normally want the sense of the lock circuit to home in on the centre frequency of the carrier component.

We can pursue the maths a little further by making justified approximations near resonance. consider the situation where the carrier is near resonance but the modulation frequency is high enough that the sidebands are not. In this case we can assume that the sidebands are totally reflected, i.e. $F(\omega \pm \Omega) \simeq -1$, this gives, neglecting the reflected carrier which is proportional to $|F(\omega)|^2 \simeq 0$

$$P_{ref} = 2P_s - 4\sqrt{P_c P_s} \operatorname{Im}[F(\omega)] \sin \Omega t \qquad (39)$$

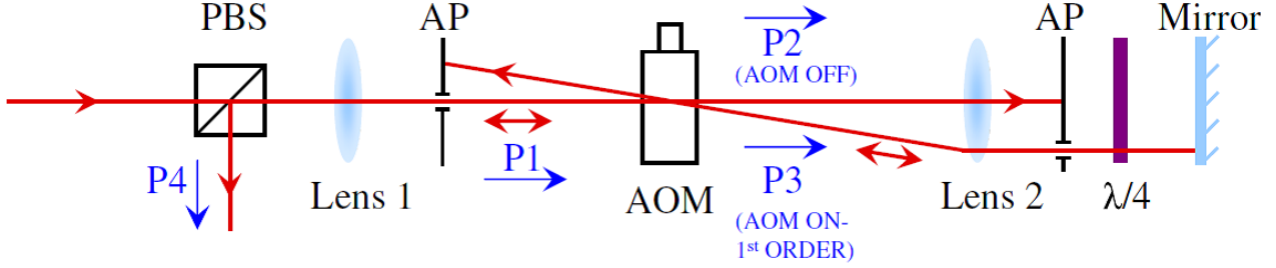On resonance ϕ will be close to a multiple of 2π so that near resonance

Figure 8: Double pass arrangement using an AOM to scan the frequency of a laser

$$\phi \simeq 2\pi N + 4\pi \times \frac{\delta L}{\lambda} \tag{40}$$

where N is an integer. Within this approximation

$$F(\delta L) \simeq \left[\frac{r}{1-r^2}\right] \times \left(i4\pi\frac{\delta L}{\lambda}\right) \sim i\frac{4\mathcal{F}}{\lambda}\delta L \tag{41}$$

where $\mathcal{F}$ is the finesse of the cavity given by

$$\mathcal{F} = \frac{\pi\sqrt{R}}{1-R} \tag{42}$$

with $r^2 = R$. The reflected power near resonance is then

$$P_{ref} \approx 2P_s - 16\sqrt{P_c P_s}\frac{\mathcal{F}}{\lambda}\delta L \sin\Omega t \tag{43}$$

which contains a term linear in δL and we are now in the linear regime, the slope of which we'll call D where

$$D \equiv 16\sqrt{P_c P_s}\frac{\mathcal{F}}{\lambda} \tag{44}$$

Other Techniques

We mention here for completeness another frequently used technique for locking lasers to reference cavities although this method does not require the use of a modulator; it makes use of frequency-dependent elliptical polarisation the details of which can be found in the source paper by Hänsch and Couillaud [2]

In essence a Brewster plate is inserted into the cavity and the back-reflected light from the cavity is analysed using a polarising beam-splitter to provide the two orthogonal linear polarisation to be detected; the difference in signals provides the error signal.

Frequency scanning using an acousto-optic modulator.

In order to stabilise both a laser and to scan smoothly the frequency by over a relatively short range (<1 GHz) an acousto-optic modulator is often used. The laser can be locked to a cavity and then the stabilised output varied in frequency by changing the rf drive frequency to the modulator. In particular the double-pass arrangement shown in figure 8 below has the benefit of providing twice the scan range without any movement of the laser beam