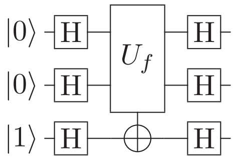
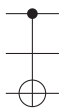
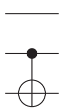
5. The Deutsch–Jozsa algorithm permits the efficient identification of classical binary functions from n bits to 1 bit which are either constant or balanced, and can be implemented in the case $n = 2$ using the network below



where the last qubit is an ancilla, and the networks below act as oracle implementations of two of the six balanced functions.

$f_{0011} =$  $f_{0101} =$ 

Draw labelled networks for oracle implementations of the four remaining balanced functions, the two constant functions, and the unbalanced function f_{0001}. [7]

Find the final state of the three qubits for the two constant functions and the six balanced functions using either matrix methods (you may find it useful to factor out the ancilla qubit and any global phases) or circuit identities, and determine the probability of finding both input qubits in the final state $|0\rangle$ in each case. Repeat this calculation for the function f_{0001} and discuss whether the Deutsch–Jozsa algorithm can be useful with unbalanced functions. [13]

Describe briefly how to implement Hadamard and CNOT gates in a trapped ion quantum computer. [5]

6. Write down the matrix form of the quantum gate ϕ_z and find the effect of applying this to a general state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. Use explicit matrix methods to show that X=HZH, and hence or otherwise find the effect of applying ϕ_x to $|\psi\rangle$. [5]

Consider an ensemble of qubits which start in the state $|\psi\rangle$ and then experience either a ϕ_z gate, an identity gate, or a ϕ_{-z} gate, chosen independently at random for each qubit in the ensemble. Show that the final state is identical to that of an ensemble of qubits which either experience a Z gate with some probability p, or are left untouched with probability $1 - p$, and find the relationship between p and ϕ. [5]

Consider the special case where $|\psi\rangle$ lies on the equator of the Bloch sphere. Calculate the purity of the final state, and find the value of ϕ which reduces the purity of the state to the minimum possible value. Use the Bloch sphere picture to explain why this occurs. [6]

Describe an error-correction network which can correct spin-flip errors using three physical qubits to encode one logical qubit. Explain why this network will also correct random ϕ_x gates, and discuss how the effectiveness varies with ϕ. How could this network be modified to correct random ϕ_z gates instead? [9]

[The purity of a density matrix ρ is defined as $\mathrm{tr}(\rho^2)$.]

7. Not all internal states of an atom are suitable for representing basis states of a qubit. State two properties of states that make a 'good' qubit. Two hyperfine states of an atom $|g\rangle$ and $|e\rangle$ with energy difference $\hbar\omega$ are used to represent a qubit. Briefly explain how coherent Rabi-oscillations can be driven between these two states. What is the effect of a resonant $\pi/2$ pulse with phase zero on the states $|g\rangle$ and $|e\rangle$? [5]

An atom is initially, at time $t = 0$, prepared in the state $|e\rangle\langle e|$, and then decays to the state $|g\rangle\langle g|$ at a rate γ. Calculate the density matrix of the atom and its entropy as a function of time and give a physical reason why this evolution cannot be unitary. What is the time evolution if the atom starts in $|g\rangle\langle g|$? The time evolution of the initial operator $|e\rangle\langle g|$ is given by

$$|e\rangle\langle g| \to \mathrm{e}^{-\mathrm{i}\omega t - \gamma t/2}|e\rangle\langle g| \; .$$

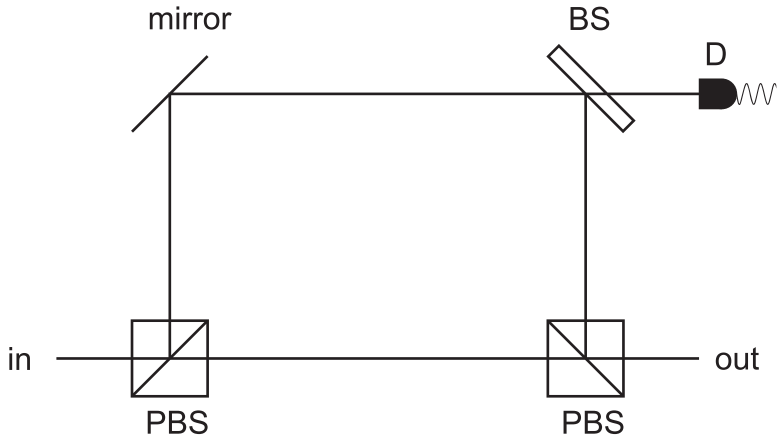
What is the evolution of the operator $|g\rangle\langle e|$?

Describe the trajectory followed by the initial density matrix $|e\rangle\langle e|$ on the Bloch sphere where the north pole corresponds to $|g\rangle\langle g|$ and the south pole to $|e\rangle\langle e|$ qualitatively as a function of t. Compare this to the trajectory of the state of an atom driven by a π pulse from $|e\rangle\langle e|$ to $|g\rangle\langle g|$. How does the entropy change along these trajectories? [10]

The atom now starts in $|g\rangle\langle g|$ and an instantaneous $\pi/2$ pulse is applied at time $t = 0$. It is then allowed to evolve for a time τ before a second instantaneous $\pi/2$ pulse is applied. The atom is measured immediately afterwards. Calculate the probability of finding the atom in state $|e\rangle$ in this measurement as a function of τ for a given ω and the cases where $\gamma = 0$ and where $\gamma \neq 0$. Briefly discuss the implications of this result for using atomic states $|g\rangle$ and $|e\rangle$ as a qubit or as the two paths of an atom interferometer. [10]

8. The computational basis states of a qubit are encoded as as horizontal $|H\rangle$ and vertical $|V\rangle$ polarizations of a photon. Which polarizations do the X and the Y basis states correspond to in this encoding? Draw schematic experimental setups using polarizing beam splitters (PBS), photo-detectors and wave-plates for measuring the qubit in each of the three bases. [5]

A two photon source produces polarization entangled photons in the state $|\alpha\rangle = \sqrt{\alpha}|VV\rangle + \sqrt{1-\alpha}|HH\rangle$ with $0 \leq \alpha \leq 1$. Calculate the joint entropy, the entropy of the reduced density matrix of each photon, and the mutual information between the photons as a function of α. Discuss the significance of the mutual information for measurements in the cases where $|\alpha\rangle$ is a product state and where it is a maximally entangled state. [8]



Alice and Bob receive one photon of $|\alpha\rangle$, respectively. Bob sends his photon through the device shown in the figure where the PBSs transmit photons in $|V\rangle$ and reflect those in state $|H\rangle$; the beam splitter (BS) has reflectivity R for both polarizations. Alice and Bob only keep those photon pairs where the detector D does not click. Work out the state of these remaining photons. How must R be chosen so that these photon pairs are maximally entangled? For which values of α is such a choice of R physically possible? If the photon source produces N pairs per second how many maximally entangled photon pairs do Alice and Bob obtain per second? [12]