5. The result of a quantum mechanical measurement can be described by a Hermitian operator M. If the system is in an eigenstate $|m\rangle$ of M, then the outcome of the measurement is the corresponding eigenvalue m (you may assume that all eigenvalues are non-degenerate), and the state is unchanged. Describe what happens if the system is not in an eigenstate. Show that the eigenvalues m are real, and that eigenstates corresponding to distinct eigenvalues are orthogonal. [6]

Consider a single qubit which is known to be in one of two states, $|0\rangle$ or $|+\rangle$, with equal probability. Explain why these states cannot be distinguished by a single measurement, and why repeated measurements do not help. Describe what happens to each of the two states if a measurement is made in the Z-basis. Show that only one of the possible outcomes can provide certain knowledge of the initial state of the qubit, while the other outcome is ambiguous. Calculate the overall probability of each outcome occurring, and determine what probabilistic conclusions can be drawn about the initial state in each case. What would happen if the measurement was made in the X-basis instead? [9]

An alternative approach, due to Helstrom, is to design a measurement which allows both states to be detected as well as possible, although neither can be detected unambiguously. In this case it is simpler to consider distinguishing between the two states

$$|a\rangle = \cos(\pi/8)\,|0\rangle + \sin(\pi/8)\,|1\rangle$$

and

$$|b\rangle = \cos(3\pi/8)\,|0\rangle + \sin(3\pi/8)\,|1\rangle,$$

which are again assumed to occur with equal probability. What probabilistic conclusions can be drawn about the initial state for each outcome of a measurement made in the Z-basis? What would happen if the measurement was made in the X-basis instead? [6]

Measuring in the Z-basis is the Helstrom optimised measurement for distinguishing between $|a\rangle$ and $|b\rangle$. Use the Bloch sphere picture to describe the states $|a\rangle$ and $|b\rangle$, and hence, or otherwise, determine the Helstrom measurement for distinguishing between $|0\rangle$ and $|+\rangle$. [4]