7. Alice and Bob share an entangled pair of qubits in the state

$$|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}.$$

Explain how this state can be used to realize the quantum dense-coding protocol. Explicitly specify a mapping of classical bit strings onto the four Bell states. Calculate the classical channel capacity achievable with this protocol by sending Alice's qubit over a quantum channel to Bob. Why does the necessary initial distribution of the entangled pair of qubits not contribute to the communication? [7]

Alice wants to communicate with Bob by sending message A with probability p_A, message B with probability p_B, and message C with probability p_C. Give an expression for the information contained in one of Alice's messages. What is the maximum amount of information she can transmit in one of her messages? Alice uses the following classical bit encoding for sending her messages to Bob

$$\begin{aligned} A &\rightarrow 0 \\ B &\rightarrow 10 \\ C &\rightarrow 11\,. \end{aligned}$$

State the conditions on the resulting bit string for this encoding to be optimal. Calculate the probabilities p_A, p_B, and p_C for the case of optimal encoding and determine the amount of information contained in one of Alice's messages in this case. Alice encodes the bit strings representing messages A, B and C into Bell states according to the mapping defined above, and transmits them via the quantum dense-coding protocol to Bob. Write down the sequence of Bell states to be transmitted for the message string $AABCACBA$. [8]

A systematic error in the creation of the initial entangled state leads to Alice and Bob carrying out the above quantum dense-coding protocol using the initial state $(2|\Phi^+\rangle + |\Phi^-\rangle)/\sqrt{5}$ with $|\Phi^-\rangle = (|00\rangle - |11\rangle)/\sqrt{2}$. What is the probability of Bob measuring a wrong Bell state? Calculate the probability that a randomly chosen bit in the classical bit string sent by Alice is received incorrectly by Bob. Does this probability depend on the chosen encoding of bit strings into Bell states? How much mutual information per bit is established between corresponding bits in Alice's and Bob's bit strings for the encoding you have chosen? [10]