Lecture 7: Optics / C2: Quantum Information and Laser Science: Quantization of the Electromagnetic Field

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Quantum optics is the study of optical phenomena that cannot be described in terms of classical electromagnetism. In order to think about these effects we need a quantum field theory for the electromagnetic field.

This is developed using the following kinds of arguments. First, recall Maxwell's equations; in the case of a dielectric with no charges or currents ($\rho_f = j_f = 0$):

$$\begin{aligned}
\nabla \times \vec{E} &= -\partial_t \vec{B} & (1)\\
\nabla \times \vec{H} &= \partial_t \vec{D} & (2)\\
\nabla \cdot \vec{D} &= 0 & (3)\\
\nabla \cdot \vec{B} &= 0 & (4)
\end{aligned}$$

The constitutive relations are taken as

$$\begin{aligned}
\vec{B} &= \mu_0 \vec{H} & (5)\\
\vec{D} &= \epsilon_0 \vec{E} + \vec{P}. & (6)
\end{aligned}$$

Now define a vector potential $\vec{A} \equiv \vec{A}(\vec{r}, t)$, such that

$$\begin{aligned}
\vec{E} &= -\partial_t \vec{A} & (7)\\
\vec{B} &= \nabla \times \vec{A} & (8)
\end{aligned}$$

Substituting these definitions in Maxwell's equations gives

$$\begin{aligned}
\nabla \times \vec{H} &= \nabla \times \vec{B}/\mu_0 = \frac{1}{\mu_0} \nabla \times \nabla \vec{A} = \frac{1}{\mu_0}\left[\nabla(\nabla \cdot \vec{A} - \nabla^2 A\right] & (9)\\
\partial_t \vec{D} &= \epsilon_0 \partial_t \vec{E} = -\epsilon_0 \partial_t^2 \vec{A} & (10)
\end{aligned}$$

Equating these and choosing $\nabla \cdot \vec{A} = 0$, we have that $\vec{A}$ satisfies a wave equation

$$\nabla^2 \vec{A} - \frac{1}{c^2} \partial_t^2 \vec{A} = 0 \quad (11)$$

The choice of $\nabla \cdot \vec{A} = 0$ is called choosing the Coulomb Guage and ensures that progagating solutions to the wave equation will be transverse.

We can solve the wave equation for $\vec{A}$ using a mode expansion as an ansatz:

$$\vec{A}(\vec{r}, t) = \sum_k \vec{u}_k(\vec{r}) A_k(t) + \vec{u}_k^*(\vec{r}) A_k^*(t) \quad (12)$$

The $\vec{u}_k(\vec{r})$'s may, for example, be plane waves, in which case:

$$\vec{u}_k(\vec{r}) = \vec{\epsilon}_k \frac{1}{\sqrt{V}} e^{i\vec{k} \cdot \vec{r}} \quad (13)$$

(Recall a similar solution for the wave equations for $\vec{E}(r,t)$ in Lt 1). Substituting this mode expansion in the wave equation gives

$$\sum_k \nabla^2 \vec{u}_k A_k + \nabla^2 \vec{u}_k^* A_k^* - \frac{1}{c^2} \vec{u}_k \partial_t^2 A_k - \frac{1}{c^2} \vec{u}_k^* \partial_t^2 A_k^* = 0 \tag{14}$$

Now <u>if</u> the $\vec{u}_k$'s solve the Helmholtz equation:

$$\nabla^2 \vec{u}_k + k^2 \vec{u}_k = 0 \tag{15}$$

Then

$$\sum_k \left(A_k + \frac{1}{c^2 k^2} \partial_t^2 A_k \right) (-k^2 \vec{u}_k) - \text{c.c.} = 0 \tag{16}$$

Therefore if each term in the sum is separately zero, the equation will be satisfied for all k, and $\vec{u}_k$. This implies

$$\partial_t^2 A_k + \omega_k^2 A_k = 0 \qquad \omega_k = ck \tag{17}$$

That is A_k's solve a harmonic oscillator-like equation. The time dependence of A_k is therefore

$$A_k(t) = \tilde{A}_k e^{i\omega_k t}. \tag{18}$$

If we solve Helmholtz equation for the $\vec{u}_k$'s using a large conducting box as a boundary condition (side L, volume $V = L^3$) then:

$$\vec{u}_k(\vec{r}) = \vec{\epsilon}_k \frac{1}{\sqrt{V}} e^{i\vec{k}\cdot\vec{r}} \qquad \int d^3 r \vec{u}_k(\vec{r}) \vec{u}_{k'}(\vec{r}) = \delta_{kk'} \tag{19}$$

with

$$\vec{k} = k_x \hat{x} + k_y \hat{y} + k_z \hat{z} \tag{20}$$

and

$$k_x = \frac{2\pi\nu}{L} \qquad k_y = \frac{2\pi\nu'}{L} \qquad k_z = \frac{2\pi\nu''}{L} \tag{21}$$

and

$$\vec{k} \cdot \vec{\epsilon}_k = 0 \tag{22}$$

The vector potential $\vec{A}(\vec{r},t)$ can then be written in terms of these functions:

$$\vec{A}(\vec{r},t) = \sum_k \vec{\epsilon}_k \frac{1}{\sqrt{V}} e^{i\vec{k}\cdot\vec{r} - \omega_k t} \tilde{A}_k + \text{c.c.} \tag{23}$$

We usually choose a slightly different normalization to make the mode amplitude coefficients dimensionless:

$$\vec{A}(\vec{r},t) = \sum_k \vec{\epsilon}_k \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}} A_k e^{i\vec{k}\cdot\vec{r} - \omega_k t} \tilde{A}_k + \text{c.c.} \tag{24}$$

Quantization of the field is accomplished now by the simple expedient of replacing the mode amplitudes (dimensionless) by operators

$$A_k \rightarrow \hat{a}_k \tag{25}$$

$$A_k^* \rightarrow \hat{a}_k^\dagger \tag{26}$$

These are called annihilation and creation operators respectively. They satisfy the bosonic commutation relation

$$[\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta_{kk'} \tag{27}$$

Now these operators should looks familiar – they are the raising and lowering operators associated with the quantum harmonic oscillator.

Recall from elementary quantum mechanics that the Hamiltonian of a 1-D harmonic oscillator is:

$$H = T + V = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \tag{28}$$

which leads to the Heisenberg equation of motion for the position operator:

$$\partial_t^2\hat{x} + \omega^2\hat{x} = 0, \tag{29}$$

hence justifying its form. The eigenfunctions of this Hamiltonian are Hermite-Gaussian functions representing the energy eigenstates. These are sketched in Fig.1.

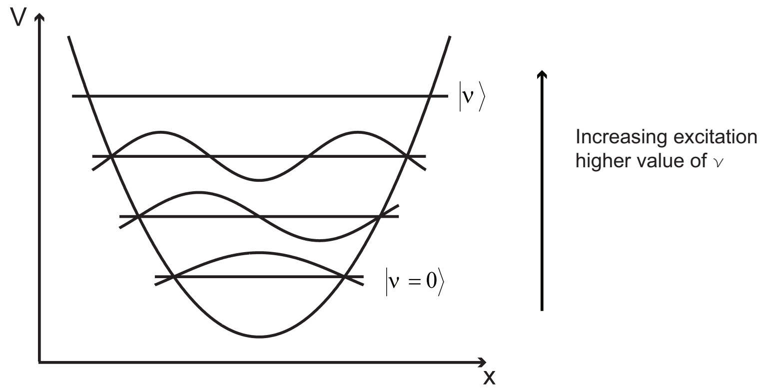


Figure 1:

The raising operator $\hat{a}^\dagger$ moves the state up the "ladder" of energy eigenstates:

$$\hat{a}^\dagger|\nu\rangle = \sqrt{\nu+1}|\nu+1\rangle \tag{30}$$

and the lowering operator steps it down

$$\hat{a}|\nu\rangle = \sqrt{\nu}|\nu-1\rangle. \tag{31}$$

The relevance to the present case of the electromagnetic field is that the local energy in the field is given by:

$$H = \int d^e r \mathcal{V} = \int d^3 r \frac{\epsilon_0}{2}\vec{E}^2 + \frac{\mu_0}{2}\vec{H}^2 \tag{32}$$

which is considered to be the Hamiltonian of the field and from which Maxwell's equation can be derived. Thus we can make the analogy that the field amplitude replace the oscillator position, as shown in Fig.2.

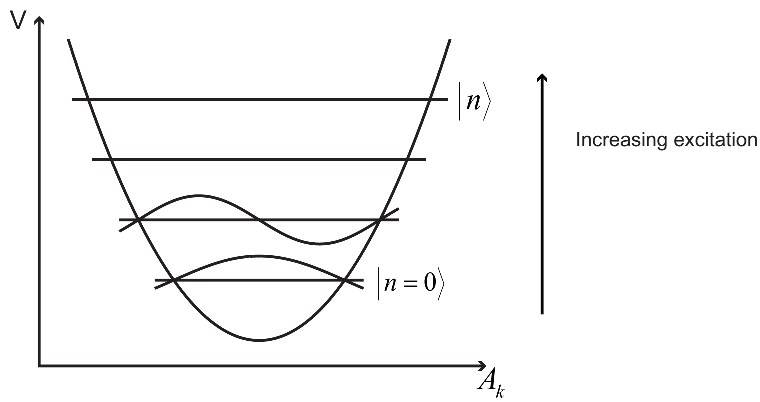


Figure 2:

Now each step up the ladder corresponds to the creation of a photon in the particular mode we are considering. The ground state of the oscillator, which has no excitation, becomes the vacuum state of the electromagnetic field, which has no photons in it.

$$|n=0\rangle = \text{no photon} \tag{33}$$

$$|1\rangle = |n=1\rangle = \hat{a}_k^\dagger|0\rangle = 1 \text{ photon} \tag{34}$$

$$|2\rangle = |n=2\rangle = \frac{\hat{a}_k^{\dagger 2}}{\sqrt{2}}|0\rangle = 2 \text{ photon} \tag{35}$$

Likewise the operator $\hat{a}_k$ takes the state down this ladder by "annihilating" photons.

It is easy to see that $\hat{n}_k = \hat{a}_k^\dagger \hat{a}_k$ gives the total number of photons in mode k of the field. The Hamiltonian may be written in terms of this operator

$$\begin{aligned} H &= \frac{1}{2}\int d^3r \epsilon_0 \vec{E}^2 + \mu \vec{H}^2 \\ &= \frac{1}{2}\int d^3r \epsilon_0 \left(\sum_k \vec{\epsilon}_k(-i\omega_k) e^{i\vec{k}\cdot\vec{r}-\omega_k t}\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}\hat{a}_k + \text{h.a.}\right)^2 \\ &\quad + \mu_0 \left(\sum_k \vec{\epsilon}_k(-i\vec{k}) e^{i\vec{k}\cdot\vec{r}-\omega_k t}\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}\hat{a}_k + \text{h.a.}\right)^2 \end{aligned} \tag{36}$$

Using the orthogonality of the mode functions in space and the dispersion relation $\omega_k = ck$, this reduces to :

$$H = \sum_k \hbar\omega_k(\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2}) \tag{37}$$

1 Nonlinear quantum optics

In a dielectric the total energy in the fields is (case without dispersion)

$$H = \int_V d^3r \vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H} \tag{38}$$

Now letting $\vec{B} = \mu_0 \vec{H}$ and $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$, we can use the usual ansatz of nonlinear optics to determine the Hamiltonian corresponding to a nonlinear optical interation. Let

$$\vec{P} = \epsilon_0 \vec{\chi}^{(1)} : \vec{E} + \epsilon_0 \vec{\chi}^{(2)} : \vec{E}\vec{E} + \cdots \tag{39}$$

Then the Hamiltonian has a linear and a nonlinear part

$$H = \underbrace{\int_V d^3r\, \epsilon_0(1+\chi^{(1)})\vec{E}\cdot\vec{E} + \mu\vec{H}\cdot\vec{H}}_{\text{linear}} + \underbrace{\int_V d^3r\, \epsilon_0 \vec{E}\cdot\vec{\chi}^{(2)} : \vec{E}\vec{E}}_{\text{nonlinear}} \tag{40}$$

The linear polarizability $\chi^{(1)}$ changes the refractive index and thus the energy density but we can quantize this part of the Hamiltonian in the standard way. Now if we consider just a $\chi^{(2)}$ of a form that yield

$$\vec{E}\cdot\vec{P}^{(2)} = \vec{E}\cdot\vec{\chi}^{(2)} : \vec{E}\vec{E} = \vec{\chi}^{(2)}\cdot(\vec{E}\cdot\vec{E})\vec{E} = \sum_{ijk}\chi^{(2)}_{ijk}E_i E_j E_k \tag{41}$$

Then we can expand the field in terms of the modes of the vector potential to obtain the nonlinear Hamiltonian

$$\begin{aligned} H &= \int_V d^3r\, \epsilon_0 \vec{\chi}^{(2)} \cdot \sum_k \sum_{k'} \sum_{k''} (-i\omega_k)(-i\omega_{k'})(-i\omega_{k''})(\vec{\epsilon}_k\cdot\vec{\epsilon}_{k'})\vec{\epsilon}_{k''}\sqrt{\frac{\hbar^3\omega_k\omega_{k'}\omega_{k''}}{8\epsilon_0^3 V^3}} \\ &\quad \times e^{i(\vec{k}+\vec{k}'+\vec{k}'')\cdot\vec{r}-i(\omega_k+\omega_{k'}+\omega_{k''})t}\hat{a}_k\hat{a}_{k'}\hat{a}_{k''} \\ &\quad + \text{lots of other terms all containing products of 3 } (\hat{a}_k, \hat{a}_k^\dagger)\text{'s} \end{aligned} \tag{42}$$

Now, because of the integrations over all space, it is clear that all terms for which $\vec{k}+\vec{k}'+\vec{k}'' \neq 0$ will result:

$$\int d^3r e^{i(\vec{k}+\vec{k}'+\vec{k}'')\cdot\vec{r}} = \delta(\vec{k}+\vec{k}'+\vec{k}''). \qquad \text{(Kronecker delta)} \tag{43}$$

Therefore all terms containing products only of creation operators or only annihilation operators will not be present. On the other hand terms like: $\hat{a}_k^\dagger \hat{a}_{k'} \hat{a}_{k''}$ will be present since the above integral is of the form:

$$\int d^3r\, e^{i(\vec{k}'+\vec{k}''-\vec{k})\cdot\vec{r}} = \delta(\vec{k}'+\vec{k}''-\vec{k}). \tag{44}$$

Satisfaction of this condition, *i.e.* $\vec{k}'+\vec{k}''-\vec{k}=0$ also implies satisfaction of the frequency condition $(\omega_k = ck)$

$$\omega_{k'} + \omega_{k''} - \omega_k = 0 \tag{45}$$

These two conditions can be thought of as the <u>momentum</u> and <u>energy</u> matching conditions for the photons, respectively.

In the case that the process conserves energy and momentum, the nonlinear part of the Hamiltonian is

$$\hat{H}_{\mathrm{NL}} = \epsilon_0 \vec{\chi}^{(2)} \cdot \sum_k \sum_{k'} \sum_{k''} (-i)\omega_k \omega_{k'} \omega_{k''} \sqrt{\frac{\hbar^3 \omega_k \omega_{k'} \omega_{k''}}{8\epsilon_0^3 V^3}} (\vec{\epsilon}_k \cdot \vec{\epsilon}_{k'}) \vec{\epsilon}_{k''} \hat{a}_{k'} \hat{a}_{k''} \hat{a}_k^\dagger + h.a. \text{ (Hermitian adjoint terms)} \tag{46}$$

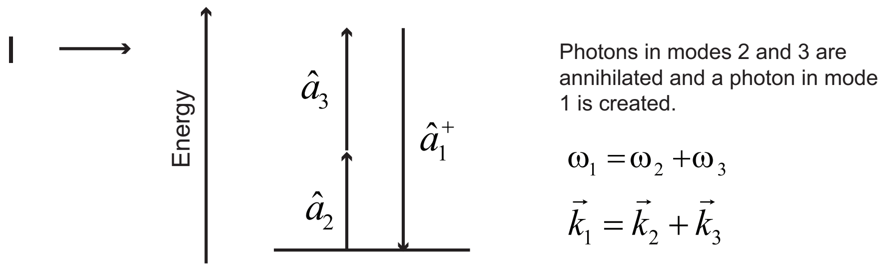
or combining all the constants:

$$\hat{H}_{\mathrm{NL}} = \sum_k \sum_{k'} \sum_{k''} \hbar g_{k'k''k} (\hat{a}_{k'} \hat{a}_{k''} \hat{a}_k^\dagger + \hat{a}_{k'}^\dagger \hat{a}_{k''}^\dagger \hat{a}_k) \tag{47}$$

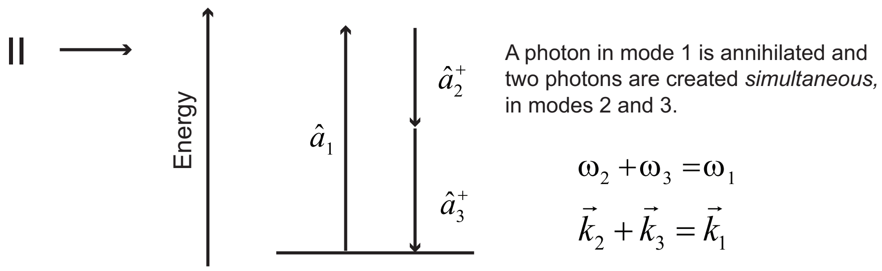
Consider the single mode case, with the lowest order nonlinear response

$$\hat{H} = \sum_{i=1,2,3} \hbar\omega_i (\hat{n}_i + \frac{1}{2}) + \hbar g (\underbrace{\hat{a}_1^\dagger \hat{a}_2 \hat{a}_3}_{\mathrm{I}} + \underbrace{\hat{a}_1 \hat{a}_2^\dagger \hat{a}_3^\dagger}_{\mathrm{II}}) \tag{48}$$

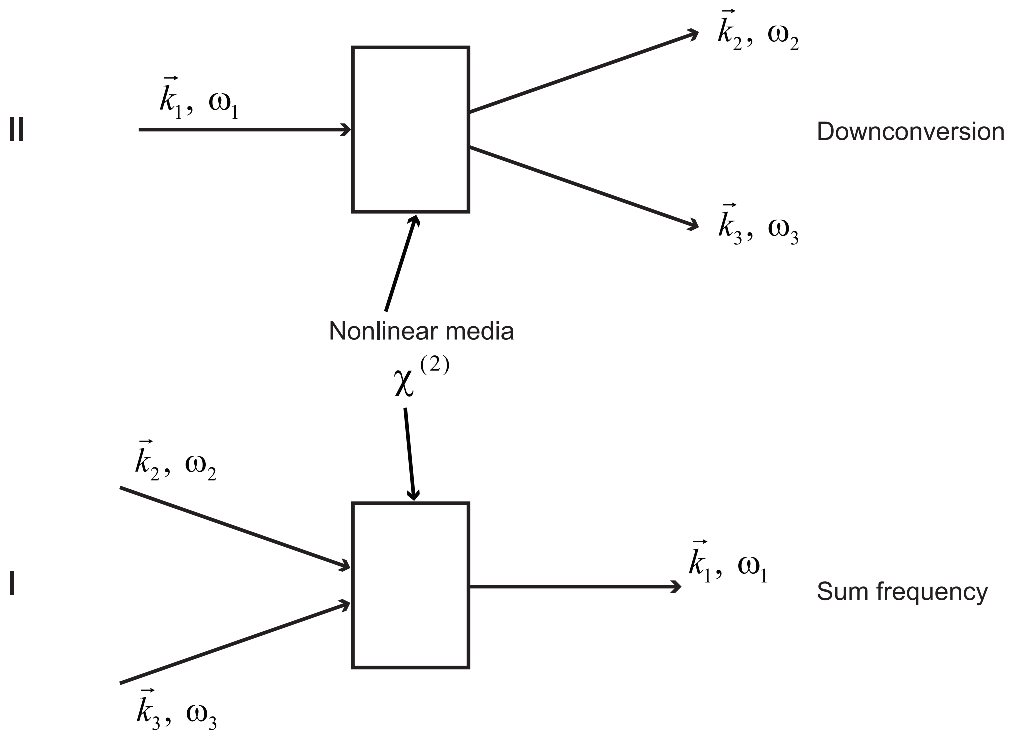
Term I in the interaction Hamiltonian describes <u>sum-frequency</u> generation.



Term II describes <u>parametric</u> downconversion.



The configuration of an experiment might be:



Note that this Hamiltonian gives the same dynamical equations for these processes as the classical field equations. To see this, we first develop Heisenberg's equation of motion for the annihilation operator

$$\begin{aligned} \dot{\hat{a}}_1 &= -\frac{i}{\hbar}[\hat{a}_1, \hat{H}] \\ &= -ig[\hat{a}_1, \hat{a}_1^\dagger \hat{a}_2 \hat{a}_3 + \hat{a}_1 \hat{a}_2^\dagger \hat{a}_3^\dagger] \end{aligned} \tag{49}$$

For simplicity let modes 2 and 3 be degenerate, so that $\hat{a}_2 = \hat{a}_3$

$$\begin{aligned} \dot{\hat{a}}_1 &= -ig[\hat{a}_1, \hat{a}_1^\dagger \hat{a}_2^2 + \hat{a}_1 \hat{a}_2^{\dagger 2}] \\ \dot{\hat{a}}_1 &= -ig\hat{a}_2^2 \end{aligned} \tag{50}$$

Similarly

$$\dot{\hat{a}}_2 = -\frac{i}{\hbar}[\hat{a}_2, \hat{H}] = -ig\hat{a}_1 \hat{a}_2^\dagger \tag{51}$$

Compare these equations for the field mode operators to those for the field amplitudes in the classical picture: (Baird notes on nonlinear optics, Eq. 19)

$$\partial_z A^{2\omega} = \kappa (A^\omega)^2 \tag{52}$$

(The change from time to space derivative can be though of in terms of a transformation to a local time frame $\tau = t - z/v$, so $\partial_t = v\partial_z$) The action of this Hamiltonian on the state of no photons (in modes 2 and 3) is to generate photons out of the vacuum ($n = 0$) state.

In the Schrödinger picture, the states evolve as

$$|\psi(t)\rangle = \hat{U}(t,0)|\psi(0)\rangle \tag{53}$$

where $\hat{U}(t,0) = e^{i\hat{H}t/\hbar}$. In the weak interaction limit; we can separate this into two components, according to the linear and nonlinear evolution:

$$\hat{U}(t,0) \approx e^{i\hat{H}_0 t/\hbar} e^{i\hat{H}_{\rm NL} t/\hbar} \approx e^{i\hat{H}_0 t/\hbar}\left(1 + \frac{i\hat{H}_{\rm NL} t}{\hbar}\right) \tag{54}$$

So the state of the system after some time is

$$|\psi(t)\rangle = |\psi(0)\rangle + \frac{i\hat{H}_{\mathrm{NL}}t}{\hbar}|\psi(0)\rangle. \tag{55}$$

Now let the initial state contain no photons in modes 2 and 3 and many in mode 1.

$$|\psi(0)\rangle = |N_1, 0_2, 0_3\rangle \tag{56}$$

So

$$\frac{i\hat{H}_{\mathrm{NL}}t}{\hbar}|\psi(0)\rangle = igt(\hat{a}_1^\dagger \hat{a}_2 \hat{a}_3 + \hat{a}_1 \hat{a}_2^\dagger \hat{a}_3^\dagger)|N_1, 0_2, 0_3\rangle. \tag{57}$$

Now the first term acts to remove photons from the vacuum and this must give a null state. The second term creates photons pairwise in modes 2 and 3.

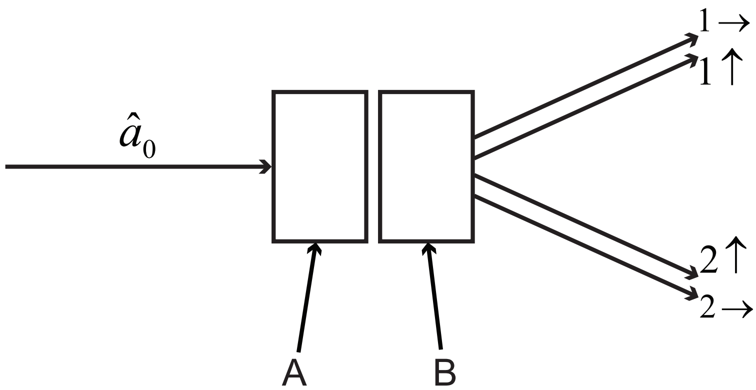
$$igt\hat{a}_1 \hat{a}_2^\dagger \hat{a}_3^\dagger |N_1, 0_2, 0_3\rangle = igt\sqrt{N_1}|N_1 - 1, 1_2, 1_3\rangle \tag{58}$$

The rate at which photon pairs are generated is proportional to the number of photons in mode 1 initially.

$$\begin{aligned} \mathrm{Prob}(1_2, 1_3) &= \text{probability of 1 photon in mode 2 and one in mode 3} \\ &= \sum_{N_1} |\langle 1_2, 1_3|\psi(t)\rangle|^2 \\ &= |gt|^2 N_1 \qquad (59) \\ \mathrm{Rate}(1_2, 1_3) &= \partial_t \mathrm{Prob}(1_2, 1_3) = 2|g|^2 t N_1 \qquad (60) \end{aligned}$$

2 Entangled photon generation

Now consider the case of two nonlinear crystals situated back to back.



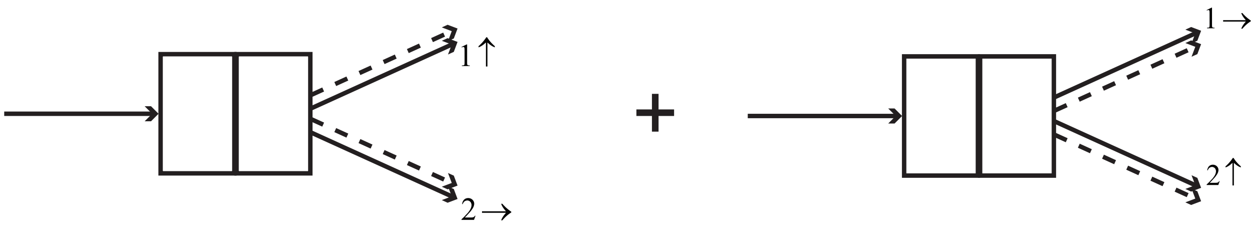
Let's say crystal A generates photon pairs in modes $1\uparrow$ and $2\rightarrow$, crystal B generates pairs in modes $1\rightarrow$ and $2\uparrow$. The total Hamiltonian is then (just the nonlinear part)

$$\begin{aligned} \hat{H} &= \hat{H}_A + \hat{H}_B \\ &= \hbar g_A \hat{a}_0 \hat{a}_{1\uparrow}^\dagger \hat{a}_{2\rightarrow}^\dagger + \hbar g_B \hat{a}_0 \hat{a}_{1\rightarrow}^\dagger \hat{a}_{2\uparrow}^\dagger + \text{h.a.} \\ &= \hbar(g_A \hat{a}_{1\uparrow}^\dagger \hat{a}_{2\rightarrow}^\dagger + g_B \hat{a}_{1\rightarrow}^\dagger \hat{a}_{2\uparrow}^\dagger)\hat{a}_0 + \text{h.a.} \qquad (61) \end{aligned}$$

It is clear that this Hamiltonian acting on the vacuum will produce correlated pair of photons in the state (taking $g_A = g_B = g$)

$$|\psi(t)\rangle = it\sqrt{N_0}g\{|N_0 - 1, 1_{1\uparrow}, 1_{2\rightarrow}\rangle + |N_0 - 1, 1_{1\rightarrow}, 1_{2\uparrow}\rangle \tag{62}$$

That is we get a pair of photons either in the upper set of 1, 2 modes or in the lower set.



According to quantum mechanics, we must add the probability amplitudes for these two events, and this gives us a quantum correlate, or entangled state. This particular example is called the triplet state of a photon pair, and is usually given the notation

$$|\psi\rangle = \frac{1}{\sqrt{2}}\{|\uparrow,\rightarrow\rangle + |\rightarrow,\uparrow\rangle\} \tag{63}$$

It has a number of very interesting properties not found in classical optics, with applications in quantum technologies such as information processing, about which you will learn in the next few lectures in this course.