Lecture 4: Optics / C2: Quantum Information and Laser Science

November 14, 2008

1 Gaussian Beam

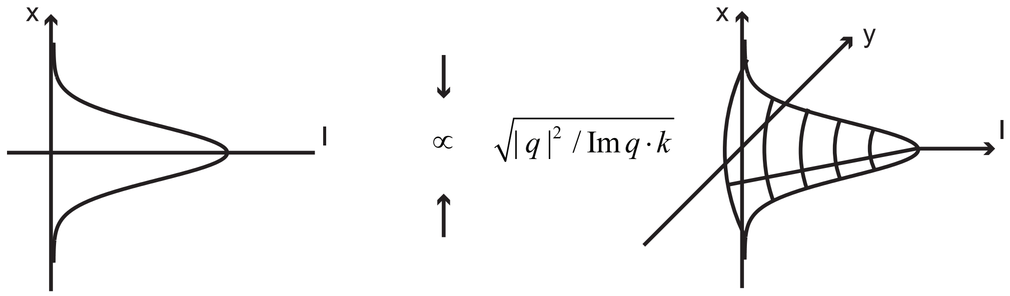
An important class of propagation problem concerns well-collimated, spatially localized beams, such as those emanating from a laser. It is therefore important to understand how these propagate through optical systems. A particularly important example of this class of beams are the so-called Gaussian beams. These take the form of a localized intensity distribution on any plane intersecting the beam, with the intensity envelope falling off as a Gaussian function of the distance away from the beam center. The simplest of these beams takes the general form described by the scalar field:

$$\varphi(x, y, 0) = \epsilon_0 \frac{e^{ik(x^2+y^2)/q^2}}{q}, \qquad q \text{ complex.} \tag{1}$$

The modulus of the field is therefore

$$|\varphi(x, y, 0)| = |\epsilon_0|^2 \frac{e^{ik(x^2+y^2)(1/2q-1/2q^*)}}{|q|^2}. \tag{2}$$

Assuming that $1/q - 1/q^* = \mathrm{Im}q/|q|^2 \neq 0$ then the intensity profile is of the form of a Gaussian, with a width proportional to $\sqrt{|q|^2/\mathrm{Im}q \cdot k}$



Also we can interpret the real part of q as a radius of curvature. For instance, if $\mathrm{Re}1/q = 1/R$, then $\varphi(x, y, 0)$ has the same form as a spherical wave (in the paraxial approximation) of radius R. Thus we can think of q as a "complex radius of curvature", describing both the curvature of the phase fronts of the wave, and the size of the beam itself. The nice thing about this form is that it is not only a reasonable approximation to a laser beam, but also it solves the Kirchoff diffraction integral in the Fresnel approximation.

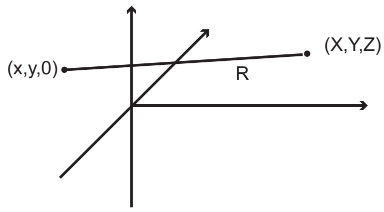
2 Propagation of Gaussian beams in free space

The Fresnel approximation to the Kirchoff diffraction integral is

$$\varphi(X, Y, Z) = \frac{1}{i\lambda} \int_\sigma dxdy\ \varphi(x, y, 0) \frac{e^{ikR}}{R} \tag{3}$$

where

$$R = \sqrt{(X-x)^2 + (Y-y)^2 + Z^2} \cong Z + \frac{(X-x)^2}{2Z} + \frac{(Y-y)^2}{2Z}. \tag{4}$$



Taking $\varphi(x, y, 0)$ as the Gaussian field given by Eq. 1, then

$$\begin{aligned}
\varphi(X,Y,Z) &= \frac{1}{i\lambda}\epsilon_0 \int_{-\infty}^{\infty} dxdy\; e^{ikZ}\, \frac{e^{ik(X-x)^2/2Z+ik(Y-y)^2/2Z}}{Z}\, \frac{e^{ik(x^2+y^2)/2q}}{q} \\
&= \frac{e^{ikZ}\epsilon_0}{i\lambda} \int_{-\infty}^{\infty} dx\; \frac{e^{ikX^2/2Z} e^{-ikXx/Z+ikx^2/2Z+ikx^2/2q}}{\sqrt{Zq}} \int_{-\infty}^{\infty} dy\; [x \to y] \\
&= e^{ikZ+ikX^2/2Z}\, \frac{\epsilon_0}{i\lambda} \left\{ \frac{1}{\sqrt{Zq}} \int_{-\infty}^{\infty} dx\; e^{\frac{ikx^2}{2}\left(\frac{1}{Z}+\frac{1}{q}\right)} e^{-ikXx/Z} \right\} \int_{-\infty}^{\infty} dy\; [x \to y]
\end{aligned} \tag{5}$$

where the $y-$integral is the same as the $x-$ integral with the replacement of variables as indicated in brackets. The integrals can be evaluated with the help of the lemma

$$\int_{-\infty}^{\infty} d\varsigma e^{-\alpha\varsigma^2+\beta\varsigma} = \sqrt{\frac{\pi}{\alpha}} e^{\beta^2/4\alpha} \qquad \mathbb{R}\mathrm{e}\alpha > 0 \tag{6}$$

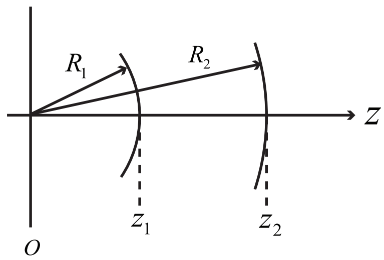
So that:

$$\begin{aligned}
\varphi(X,Y,Z) &= e^{ikZ+ikX^2/2Z}\, \frac{\epsilon_0}{i\lambda} \left\{ \frac{1}{\sqrt{Zq}} e^{-\frac{k^2X^2}{Z^2}\left(\frac{1}{-4i\frac{k}{2}\left(\frac{1}{Z}+\frac{1}{q}\right)}\right)} \sqrt{\frac{\pi}{\frac{k}{2}\left(\frac{1}{Z}+\frac{1}{q}\right)}} \right\} \int dy\; [x \to y] \\
&= e^{ikZ}\, \frac{\epsilon_0}{i\lambda} \left\{ \sqrt{\frac{\lambda}{q+Z}} e^{i\frac{ikX^2}{2}\left(\frac{1}{Z}-\frac{1}{Z+Z^2/a}\right)} \right\} \int dy\; [x \to y]
\end{aligned} \tag{7}$$

After same simple rearrangements:

$$\varphi(X,Y,Z) = e^{ikZ}\epsilon_0 \frac{e^{ik(X^2+Y^2)/q'^2}}{q'} \tag{8}$$

with $q' = q + Z$. That is, after propagation over a distance Z, the beam remains a Gaussian, but with a

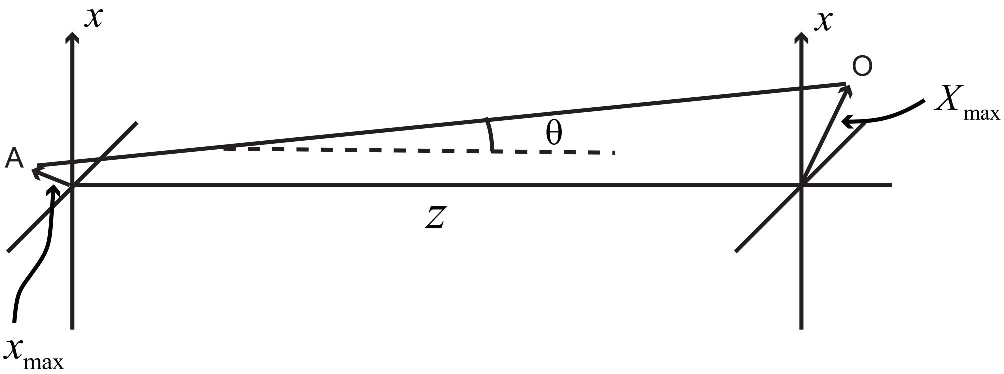


new "q-parameter" given by $q' = q + Z$. We can interpret this in light of our conjecture that $\mathbb{R}\mathrm{e}1/q = 1/R$ is an effective curvature of the phase front of the Gaussian beam. For example, if we consider a spherical wave emanating from a point $z = 0$, then the radius of curvature of the wavefront a distance z_1 away is

$R_1 = z_1$, and a distance z_2 away is $R_2 = z_2 = R_1 + (z_2 - z_1)$. This gives us an important clue as to how to generalize the beam propagation to more complicated optical systems, including interfaces, mirrors, lenses, *etc.* The idea is that we know how to track changes in the radius of curvature of a wavefront using the rules of paraxial optics. In order to see this, it is useful to remind ourselves of some of the rules of paraxial ray tracing, in particular the matrix method in terms of transfer functions.

3 Paraxial ray tracing

Paraxial optics is defined by the notion that everything of interest takes place "close" to the optical axis (usually a rotational symmetry axis) of the optical system. In the Kirchoff diffraction integral it is defined by the requirement that $x_{max}/Z \ll 1$, where x_{max} is the largest value of the transverse coordinate in the aperture or observation place and Z is the distance between the aperture and observation places.



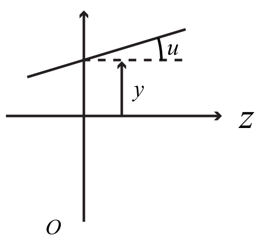
It is clear from the figure that if θ is the angle of the line $[OA]$ with respect to the z-axis, then

$$\tan\theta \le \frac{x_{max} + X_{max}}{Z} \ll 1, \tag{9}$$

so that we can think of the paraxial approximation as holding for all optical paths $[OA]$ (i.e. for all rays), such that

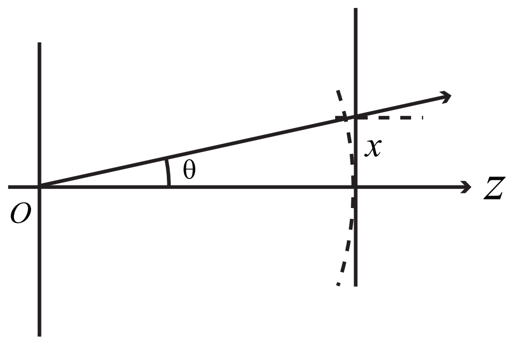
$$\tan\theta \approx \sin\theta \approx \theta = \frac{X}{Z}. \tag{10}$$

In fact in paraxial optics, rays are characterized by $[x, \theta]$, the distance of the ray from the axis and its angle with respect to that same axis, all defined at a give plane. It is usual to consider only 2-D propagation and (for reasons that will become clear) to include the refractive index, n, of the medium in which the ray is propagating in the angle parameter. Refering to the figure, the ray at place O is then specified by the pair $[y, nu]$, as shown in the figure.



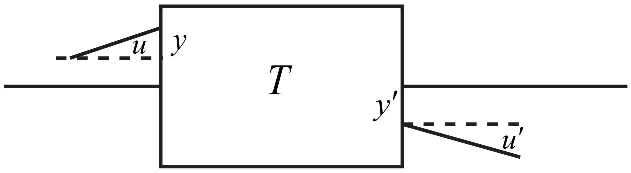
Paraxial ray tracing deals with how these parameters change as the ray propagates through a system of optical elements. Note that these parameters can be related back to those of a spherical wave in the following way. Consider a point source S. The radius of curvature of the wave emanating from S is $Z = x/\theta$, where Z is the distance from the source. Thus we can derive the curvature of a wavefront from a knowledge of the paraxial ray parameters:

$$R = \frac{y}{u} \tag{11}$$



4 Ray Transfer Matrix, $\underline{\underline{T}}$

The changes in $[y, nu]$ are calculated using the *ray transfer matrix* $\underline{\underline{T}}$ that characterizes a paraxial system. Let $[y', n'u']$ be the ray parameters after the optical element characterized by the 2×2 matrix T

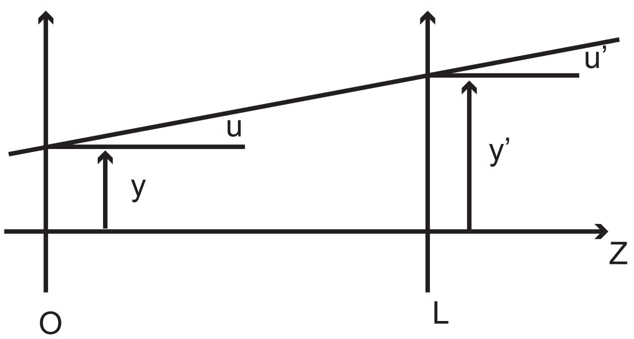


with elements A, B, C, D. These are defined by the relation

$$\begin{bmatrix} y' \\ n'u' \end{bmatrix} = \underline{\underline{T}} \begin{bmatrix} y \\ nu \end{bmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{bmatrix} y \\ nu \end{bmatrix} \tag{12}$$

$\underline{\underline{T}}$ can be determined for a few simple and important optical elements.

1. <u>Free space propagation</u>

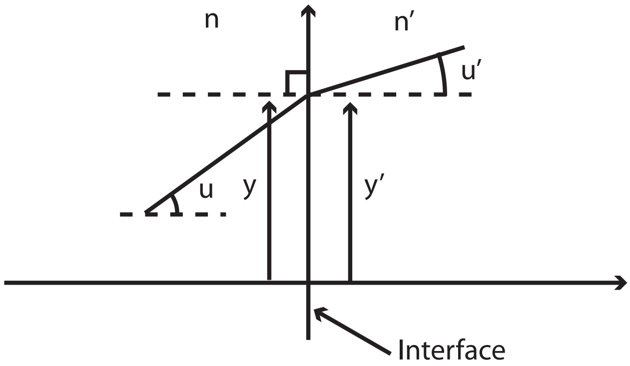


By inspection:

$$\begin{aligned} y' &= y + Lu = y + \frac{L}{n} nu \\ u' &= u \\ n'u' &= n'u = nu \end{aligned} \tag{13}$$

(assuming the "output" is just before any change in medium).

$$\therefore \quad \underline{\underline{T}}_F = \begin{pmatrix} 1 & L/n \\ 0 & 1 \end{pmatrix} \tag{14}$$



2. Refraction at a plane surface

   By inspection, since the "input" is just to the left of the surface and the "output" just to the right:

$$y' = y. \tag{15}$$

   Refraction at the surface is governed by Snell's law:

$$n' \sin i' = n \sin i, \tag{16}$$

   which is, in the paraxial approximation

$$n'i' = ni. \tag{17}$$

   But for plane interface:

$$i' = u' \tag{18}$$
$$i = u, \tag{19}$$

   so that

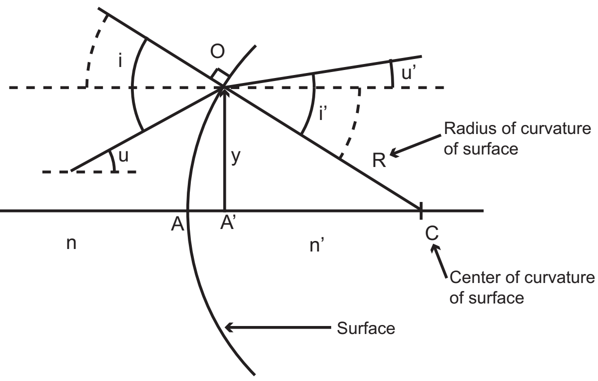
$$n'u' = nu, \tag{20}$$

   and the transfer matrix is:

$$\underline{\underline{T}}_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underline{\underline{1}} \qquad (2 \times 2 \text{ identity}) \tag{21}$$

   The simple form of this transfer matrix is because of the choice of nu rather than simply u as the variable specifying the ray direction.

3. Refraction at a curved surface



   By symmetry:

$$i' - u' = i - u \tag{22}$$

By definition in $\triangle OCA'$:

$$\tan(i' - u') = \frac{y}{R} \tag{23}$$

or, in the paraxial approximation:

$$i' - u' = y/R \tag{24}$$

So that

$$u' = i' - y/R \tag{25}$$

Then

$$n'u' = n'i' - n'y/R \tag{26}$$

Using the paraxial from of Snell's law

$$n'u' = ni - n'y/R \tag{27}$$

and

$$\begin{aligned} n'u' &= n(u + y/R) - n'y/R \\ &= nu + \frac{n - n'}{R} y \\ &= nu - \frac{n' - n}{R} y \\ n'u' &= nu - \phi y, \end{aligned} \tag{28}$$

where $\phi = (n' - n)/R$ is called the *power* of the surface.

Again using the idea that the input and output places are just to the left and right of the surface, and that in the paraxial approximation A and A' are almost coincident,

$$y = y', \tag{29}$$

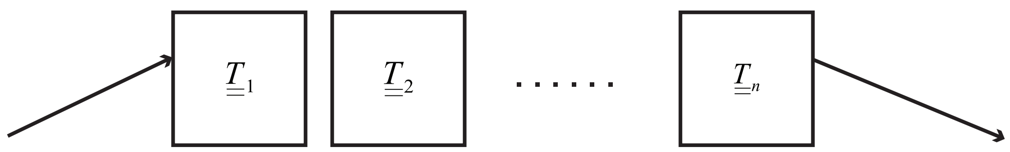
so that

$$\underline{\underline{T}}_s = \begin{pmatrix} 1 & 0 \\ -\phi & 1 \end{pmatrix}. \tag{30}$$

In general any optical system with power (i.e. refracting capability), such as a lens, follows this same formula. If the lens has focal length f, then

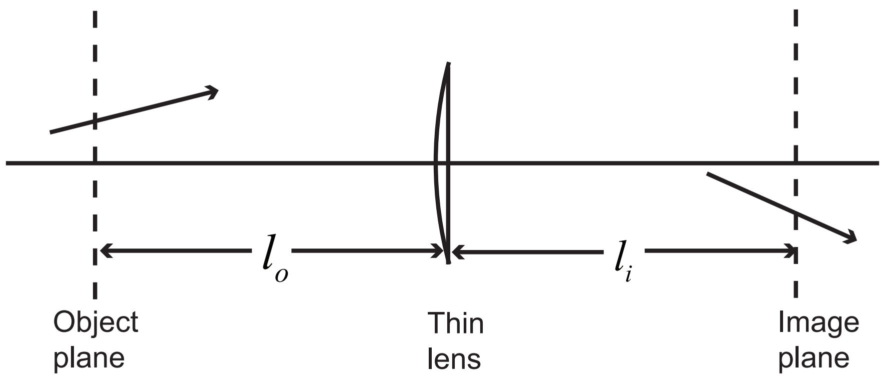
$$\underline{\underline{T}}_l = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \tag{31}$$

The transfer matrix for an optical system consisting several elements is found by concatenating a sequence of individual elementary transfer matrices.



$$\underline{\underline{T}}_{Total} = \underline{\underline{T}}_n \cdots \underline{\underline{T}}_3 \times \underline{\underline{T}}_2 \times \underline{\underline{T}}_1 \tag{32}$$

Note that is important that the matrices are arranged from right to left in the order in which the ray ...... them. This is because the effect of the elements depends on their ordering, and this is reflected in the non-commutativity of the matrices. Thus an imaging system using a simple thin lens has the transfer matrix $\underline{\underline{T}}_{is}$:



$$\underline{\underline{T}}_{is} = \begin{pmatrix} 1 & l_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & l_o \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 - l_i/f & l_o + l_i - l_i l_o/f \\ -1/f & 1 - l_o/f \end{pmatrix} \tag{33}$$

The standard lensmakers formula (or Newton's equation) can be found easily from this. Consider a ray leaving the object plane at zero height, so that $y = 0$. It will reach the image plane at the same height, $y' = 0$, since this is the definition of the image plane conjugate to our chosen object plane. This implies that the B element of $\underline{\underline{T}}_{is}$ must be zero:

$$l_o + l_i - \frac{l_o l_i}{f} = 0 \tag{34}$$

So that this is given by the formula

$$\frac{1}{l_o} + \frac{1}{l_i} = \frac{1}{f} \tag{35}$$

5 Application of $\underline{\underline{T}}$ to Gaussian beam propagation

We are interested in how the q-parameter changes. We have seen that this is something like a complex radius of curvature of the wavefront of the Gaussian beam, so let us consider first how the radius of curvature of a spherical (paraxial) wave changes.
We have

$$R \cong y/u \tag{36}$$

and

$$R' \cong y'/u' \tag{37}$$

for the radius of curvature of the wavefronts at two planes. If these planes are seperated by optical elements with ray transfer matrix $\underline{\underline{T}}$. Then

$$\begin{pmatrix} y' \\ n'u' \end{pmatrix} = \underline{\underline{T}} \begin{pmatrix} y \\ nu \end{pmatrix} = \begin{pmatrix} Ay + Bnu \\ Cy + Dnu \end{pmatrix} \tag{38}$$

Thus, taking the ratio of the two equations encapsulated in these matrices:

$$\frac{y'}{n'u'} = \frac{Ay + Bnu}{Cy + Dnu}$$
$$= \frac{A\frac{y}{nu} + B}{C\frac{y}{nu} + D} \tag{39}$$

Using the approximations for the radii of curvature

$$R'/u' \approx \frac{AR/u + B}{CR/u + D}. \tag{40}$$

Finally we replace R/u by q, so that

$$q' = \frac{Aq + B}{Cq + D}. \tag{41}$$

This last step is of course, merely a conjecture. We have seen, however, that is it true for at least one case - free space propagation - and it is easy to prove the same for the action of thin paraxial lens. The leap proposed here though is that the formula is general, and that it relates the q-parameter of the Gaussian beams at the input and output planes of an arbitrary paraxial optical systems. The recipe for propagating beams is thus:

1. use paraxial ray tracing to find the $\underline{\underline{T}}$ matrix of the systems,
2. use Eq. 38 to find the q-parameter after the system, given the input q parameter.

As we shall see, this formula is particularly important in finding the stability condition of a complex laser cavity.

6 Properties of Gaussian beams

The fundamental Gaussian beam is characterized by a complex parameter q. In order to make sense of the propagation of such an beam, we have identified two important properties;

1. q must have a non-zero imaginary component
2. The real part of q acts like a radius of curvature of the wavefront at the reference plane location.

Thus, without loss of generality, we can write

$$q = z - ib \tag{42}$$

where $z =$ distance from the reference plane, and $b > 0$. Inserting this into the Gaussian beam formula we find

$$\Phi(x, y, 0) = \epsilon_0 \frac{e^{ik(x^2+y^2)/2q}}{q}, \tag{43}$$

with

$$\frac{ik}{2q} = \frac{ik}{2(z - ib)} = \frac{ik(z + ib)}{2(z^2 + b^2)} = \frac{ikz}{2(z^2 + b^2)} - \frac{kb}{2(z^2 + b^2)}. \tag{44}$$

In terms of a radius of curvature R, and a beam size w:

$$\frac{ik}{2q} = \frac{ik}{2R} - \frac{1}{w^2} \tag{45}$$

the beam equation becomes

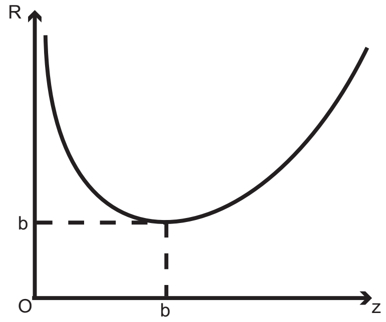
$$\Phi(x, y, 0) = \frac{\epsilon_0}{q} e^{ik(x^2+y^2)/2R} e^{-(x^2+y^2)/w^2} \tag{46}$$

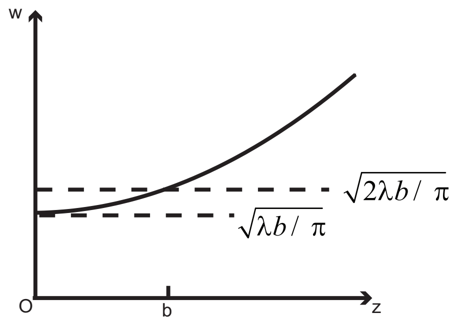
The first exponential term can be recognized as the paraxial approximation to a spherical wave, the second as a Gaussian amplitude function. The radius of curvature is

$$R(z) = \frac{z^2 + b^2}{z} = z + \frac{b^2}{z} \tag{47}$$

and the beam size

$$w^2(z) = \frac{2(z^2 + b^2)}{kb} = 2\frac{\lambda b}{2\pi}\left(1 + \frac{z^2}{b^2}\right) \tag{48}$$





The parameter b is called the Rayleigh distance and is related to the sizeof the beam waist $w_0 = w(0)$

$$w(0) = \sqrt{\frac{\lambda b}{\pi}} \tag{49}$$

$$b = \frac{\pi w^2(0)}{\lambda} \tag{50}$$

i.e, it is roughly the area of beam at the waist divided by the wavelength. The Rayleigh distance is that propagation distance over which the beam doubles in size. This says that small beams (small w_0) diffract quickly, since they have a small Rayleigh distance.
The term $1/q$ may also be written in terms of its real and imaginary components

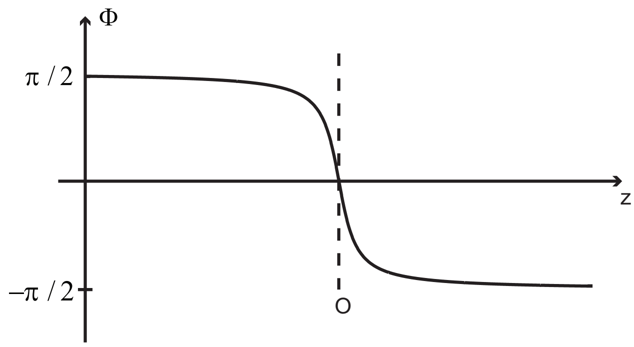
$$\frac{1}{q} = \frac{1}{z - ib} = \frac{z + ib}{z^2 + b^2} = \frac{\sqrt{z^2 + b^2}}{z^2 + b^2} e^{-i \tan^{-1} z/b} = \frac{i}{b} \frac{w(0)}{w(z)} e^{-i \tan^{-1} z/b}. \tag{51}$$

Note the important features of this equation:

1. The prefactor contains a weight $w(0)/w(z)$. Thus it decreases as the beam propagates way from the waist. This prefactor ensures the decrease in wave amplitude just compensates the increase in spot size in such a way that the total energy in the beam remains constant.

2. There is a phase shift of π as the beam passes through its waist.

$$\Phi(z) = -\tan^{-1}(z/b) \tag{52}$$

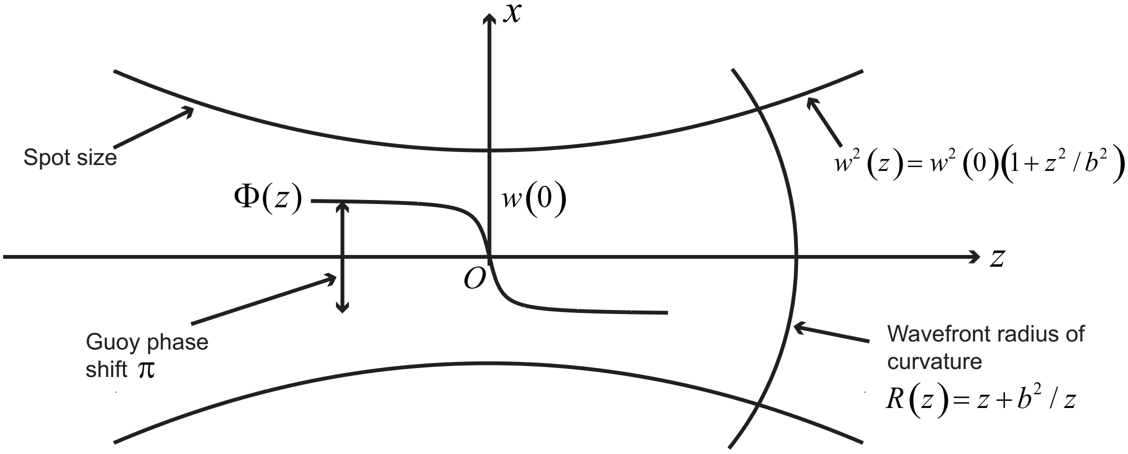


This is called the Guoy phase shift and is something that happens in all beams as they pass through a focus, not just Gaussian ones. It is especially important in nonlinear optics, where it can affect the phase matching condition, as we shall see.

The overall field for a Gaussian beam therefore has the form:

$$\Phi(x, y, z) = \epsilon_0 \frac{i}{b} \frac{w(0)}{w(z)} e^{-i \tan^{-1} z/b} e^{-(x^2+y^2)/w^2(z)} e^{ik(x^2+y^2)/2R} \tag{53}$$

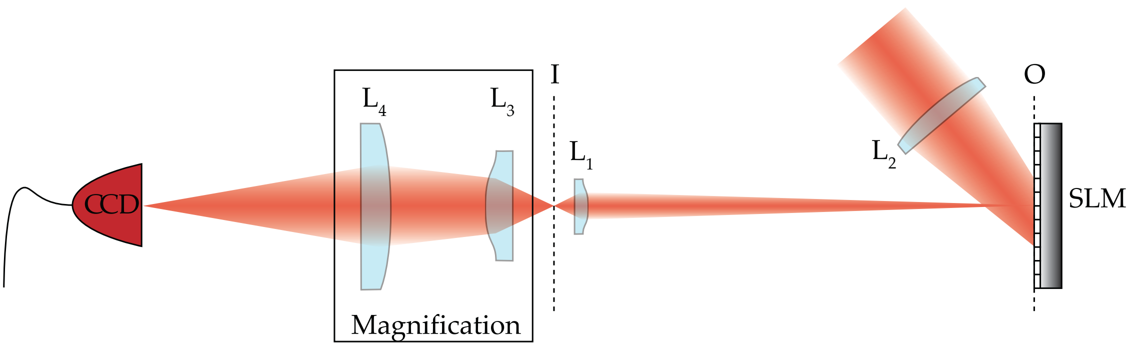
The parameters can be represented graphically



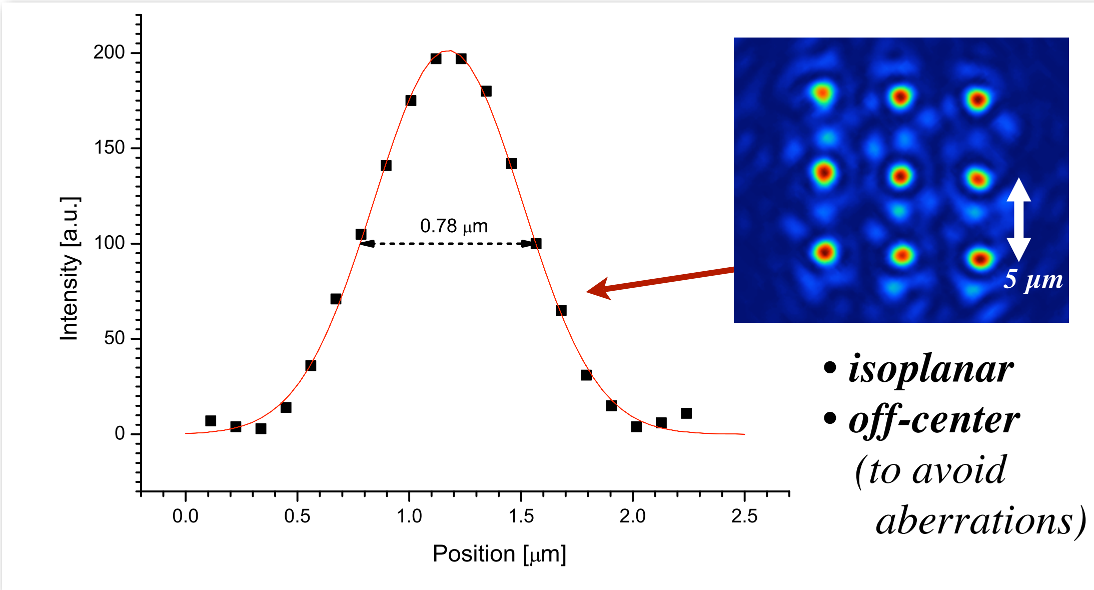
We have dealt here with the simplest possible Gaussian beam. There are more complicated forms, related to the family of Hermite-Gauss polynomials, as well as other families with similar properties, such as the radially symmetric Gauss-Laguerre beams. These are covered in a number of texts, such as Siegman.

7 Application of Gaussian beam propagation to atom trapping

Laser beams can trap atoms and other polarizable particles. An example of this is the use of a spatial-light modulator (SLM), consisting of an array of mirrors, that generate a regular array of "spots" (i.e. focussed beams) by reflecting parts of an input Gaussian beam from a laser. The imaging set up is shown in the figure, and makes use of the Fresnel diffraction configuration set ups that we discussed in the last lecture, in particular those that have no field-angle dependent phase. In this arrangement, the SLM in the object



plane acts like an array of apertures, each of size 12.5x12.5μm^2.. Mirrors tilted to the "on" position reflect light into L1. Light reflected by mirrors in the "off" position just misses L1. L2 is a lens of diameter D=40mm, and focal length f=750mm, positioned so that the illuminating light passes through the centre of L1 (if the SLM mirrors are all in the "on" position), which is a focussing aspheric lens (D=25mm; f=20mm, NA=0.5) at a distance of about 650mm from the SLM . I is the image plane, where the SLM image is de-magnified by a factor M=32. L3 and L4 were used to re-magnify the image for observation with a CCD camera. Their NA is larger than the one of L1, so that the re-magnification does not affect the image. This gives rise to an array of spots that are on the order of 50μm in diameter. Each of these spots can act as a dipole trap for a single atom, and the array of these trapped atoms can then be manipulated by



detecting photons emitted from them or by bringing them close to one another, in a controlled collision, in such as way as to entangle their quantum states. The array of trapped single atoms acts as a cluster of individual qubits, and their interactions engenders a cluster state for one-way quantum computing.