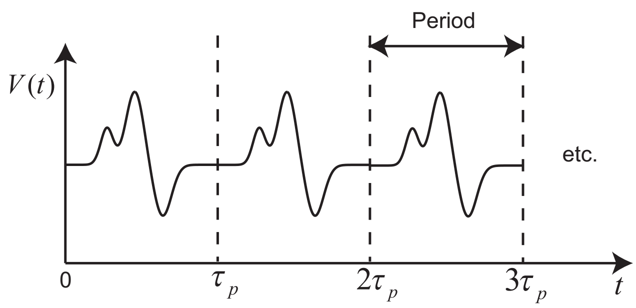
Lecture 2: Optics / C2: Quantum Information and Laser Science

October 29, 2008

1 Fourier analysis

This branch of analysis is extremely useful in dealing with linear systems (*e.g.* Maxwell's equations for the most part), when we want to go beyond plane wave with monochromatic frequency. The basic idea is that any periodic signal, $V(t)$ (say a scalar at present), with period τ_p, can be represented as a sum of sines and consines with discrete frequencies. Let



$$V(t) = \sum_{n=0}^{\infty} a_n \cos(n\omega_p t + \alpha_n) \tag{1}$$

where $\omega_p = 2 * \pi/\tau_p$, a_n = amplitude of component at frequency $n\omega_p$, α_n = phase of the component at frequency $n\omega_p$. Then the set of real numbers $\{\alpha_n, a_n\}$ completely specify the signal, once τ_p is specified. They can be found by using the orthogonality properties of the series:

$$\begin{aligned} V_n &= \frac{1}{\tau_p}\int_0^{\tau_p} dt\, V(t)e^{in\omega_p t} = \sum_{n'=0}^{\infty}\frac{1}{\tau_p}\int_0^{\tau_p} dt'\, a_{n'}\cos(n'\omega t + \alpha_{n'})e^{in\omega_p t} \\ V_n &= \sum_{n'=0}^{\infty} A_{n',n} \end{aligned} \tag{2}$$

where

$$\begin{aligned} A_{n',n} &= \frac{1}{\tau_p}\int_0^{\tau_p} dt\, \frac{a_{n'}}{2}e^{i(n+n')\omega_p t + i\alpha_{n'}} + \frac{a_{n'}}{2}e^{-i(n-n')\omega_p t - i\alpha_{n'}} \\ &= \frac{a_{n'}e^{i\alpha_{n'}}}{2\tau_p}\left[\frac{e^{i(n'+n)\omega_p t}}{i(n'+n)\omega_p}\right]_0^{\tau_p} + \frac{a_{n'}e^{-i\alpha_{n'}}}{2\tau_p}\left[\frac{e^{-i(n'-n)\omega_p t}}{-i(n'-n)\omega_p}\right]_0^{\tau_p} \\ &= \frac{a_{n'}e^{i\alpha_{n'}}}{2\tau_p}\delta_{n',n} + \frac{a_{n'}e^{-i\alpha_{n'}}}{2\tau_p}\delta_{n',n} \end{aligned} \tag{3}$$

where the Kronecker-delta symbol is defined by:

$$\begin{aligned} \delta_{n,m} &= 1 \qquad n = m \\ &= 0 \qquad \text{otherwise.} \end{aligned} \tag{4}$$

Then the coefficients are;

$$\begin{array}{llll} n > 0 & V_n & = & \dfrac{\alpha_n e^{i\alpha_n}}{2} \\ n < 0 & V_n & = & \dfrac{\alpha_n e^{-i\alpha_n}}{2} \\ n = 0 & V_0 & = & a_0 \end{array} \tag{5}$$

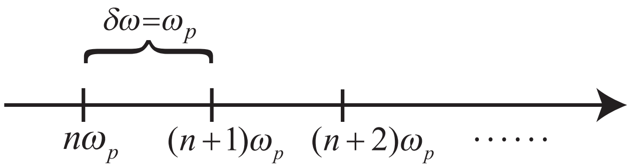
and in general $V_n^* = V_{-n}$. The function $V(t)$ can therefore by expressed as

$$V(t) = \sum_{n=-\infty}^{\infty} V_n e^{-in\omega_p t} \qquad V_n = \frac{1}{\tau_p} \int_0^{\tau_p} dt\, V(t) e^{in\omega_p t} \tag{6}$$

2 Fourier Transforms

If a function $V(t)$ is non-periodic, it can not be expanded in a Fourier series. This is because $\tau_p \to \infty$, and the sum becomes an integral. Recall that for a periodic function:

$$V(t) = \sum_{n=-\infty}^{\infty} V_n e^{-in\omega_p t} \tag{7}$$



Define a continuous function $\tilde{V}(\omega)$ such that the coefficients V_n are samples of this function

$$A_n = \tilde{V}(n\delta\omega)\frac{\delta\omega}{2\pi}. \tag{8}$$

Then

$$V(t) = \sum_{n=-\infty}^{\infty} \frac{\delta\omega}{2\pi}\tilde{V}(n\delta\omega)e^{in\delta\omega t} \tag{9}$$

Now take the limit $\delta\omega \to 0$, with $n\delta\omega \to \omega$

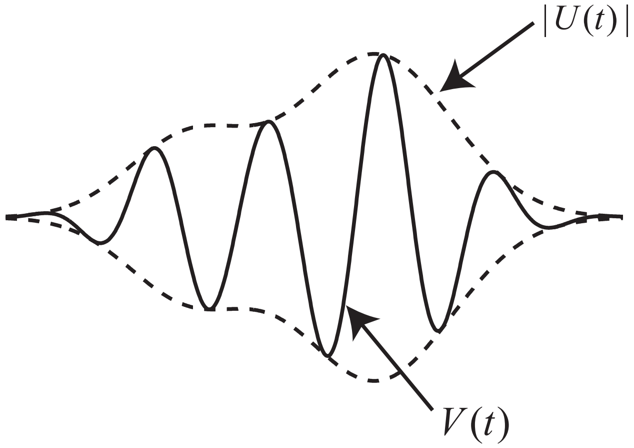
$$V(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \tilde{V}(\omega)e^{i\omega t} \tag{10}$$

This defines the function $V(t)$ as the Fourier Transform of a conjugate fuction $\tilde{V}(\omega)$. Since $V(t)$ is real, $V(t) = V^*(t)$, so $\tilde{V}(\omega) = \tilde{(V)}^*(-\omega)$, which is analogous to the property of the Fourier coefficients $V_n = V_{-n}^*$ in the series expansion.

As we have seen, it is often convenient to work with a complex signal field, rather than a real one; and it is useful to define the <u>analytic signal</u>, which is the Fourier Transform of the half spectrum of $\tilde{V}(\omega)$ associated with positive frequencies. By definition

$$\begin{aligned} V(t) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \tilde{V}(\omega)e^{i\omega t} \\ &= \int_{-\infty}^{0} \frac{d\omega}{2\pi}\, \tilde{V}(\omega)e^{i\omega t} + \int_{0}^{\infty} \frac{d\omega}{2\pi}\, \tilde{V}(\omega)e^{i\omega t} \\ &= \frac{1}{2}U^*(t) + \frac{1}{2}U(t) \end{aligned} \tag{11}$$

The function $U(t)$ is the analytic signal associated with $V(t)$. Its real and imaginary parts are the cosine and sine transforms of the spectrum $\tilde{V}(\omega) = a(\omega)e^{i\alpha(\omega)}$.
It is useful, not only because it makes the mathematics more compact, but also because it is closely related to the envelope of the real signal.



The analytic signal is a convenient tool for calculating properties of non-monochromatic fields such as Poynting vector modulus or the intensity:

$$I = \frac{1}{2T}\int_{-T}^{T} dt\,(E(t)H(t)) = \frac{n}{Z_0}\frac{1}{2T}\int_{-T}^{T} dt\,E^2(t) \tag{12}$$

Now take $V(t)$ to be an aperiodic signal signal, with non-zero value only in some 'small' range of t, and let $T \to \infty$; in the integral (so long as $T >$ range of support of $V(t)$)

$$\begin{aligned} I &= \frac{n}{Z_0}\frac{1}{2T}\int_{-\infty}^{\infty} dt\,V^2(t) \\ I &= \frac{n}{Z_0}\frac{1}{2T}\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\,|\tilde{V}(\omega)|^2 = \frac{n}{Z_0}\frac{1}{T}\int_{0}^{\infty} \frac{d\omega}{2\pi}\,|\tilde{V}(\omega)|^2. \end{aligned} \tag{13}$$

It makes sense from the point of view of conservation of energy that the intensity can be expressed in the same form for both time-domain and frequency-domain signals.
Thus the power density per unit frequency interval $P(\omega)$ is defined by

$$\int_0^{\infty} P(\omega)d\omega = I \tag{14}$$

so

$$P(\omega) = \frac{n}{Z_0}\frac{1}{2\pi}|\tilde{V}(\omega)|^2 \tag{15}$$

3 Some important properties of Fourier transforms ($\mathcal{F} \equiv$ Transformation operation)

1. Linearity:

$$\mathcal{F}\{\alpha g + \beta h\} = \alpha\mathcal{F}\{g\} + \beta\mathcal{F}\{h\} \tag{16}$$

   The F.T. of the sum of two functions is the sum fo the transforms of each.

2. Similarity: If $\mathcal{F}\{g(t)\} = G(\omega)$, then:

$$\mathcal{F}\{g(at)\} = \frac{1}{|a|}G\left(\frac{\omega}{a}\right) \tag{17}$$

A scale change in ... leads to the inverse change in the conjugate domain, *e.g.* compressing the time domain ($a < 1$) expands the spectrum.

3. Shift Theorem: If $\mathcal{F}\{g(t)\} = G(\omega)$ then:

$$\mathcal{F}\{g(t-a)\} = G(\omega)e^{i\omega a} \tag{18}$$

4. Parseval's Theorem:

$$\int dt\, |g(t)|^2 = \int \frac{d\omega}{2\pi} |G(\omega)|^2 \tag{19}$$

The total power in the signal is the same in both domains.

5. Convolution Theorem: If $\mathcal{F}\{g(t)\} = G(\omega)$ and $\mathcal{F}\{h(t)\} = H(\omega)$, then

$$\mathcal{F}\left\{\int dt'\, g(t')h(t-t')\right\} = G(\omega) \times H(\omega) \tag{20}$$

Compact notation:

$$\mathcal{F}\left\{g(t) * h(t)\right\} = G(\omega) \times H(\omega) \tag{21}$$

6. Autocorrelation Theorem: (special case of Convolution Theorem)

$$\mathcal{F}\left\{\int dt'\, g(t')g(t-t')\right\} = |G(\omega)|^2 \tag{22}$$

This is closely related to the measurement of spectra of optical fields.

7. Fourier Integral Theorem

$$\mathcal{F}\mathcal{F}^{-1}\{g(t)\} = \mathcal{F}^{-1}\mathcal{F}\{g(t)\} = g(t) \tag{23}$$

The Fourier transformation operation has an inverse, except at points of discontinuity.

4 Generalized Functions: the Dirac δ function

The familiar idea of the Kronecker-delta symbol can be extended to continuous indices:

$$\text{Kronecker-delta:} \quad \begin{aligned} \delta_{nn'} &= 1 \quad \text{if } n = n' \\ &= 0 \quad \text{otherwise} \end{aligned} \tag{24}$$

This function arose naturally in the development of Fourier series. For example, it is obvious that the Fourier coefficients for the sinusoidal function:

$$V(t) = \cos(m\omega_p t) \tag{25}$$

are

$$\begin{aligned} V_n &= \frac{1}{2} \quad n = m,\, n = -m \\ &= 0 \quad \text{otherwise} \end{aligned} \tag{26}$$

or

$$V_n = \frac{1}{2}\delta_{nm} + \frac{1}{2}\delta_{n,-m}. \tag{27}$$

Now define a density function $\delta(\omega)$, such that:

$$\int_{0-}^{0+} \delta(\omega)\, d\omega = 1 \tag{28}$$

with the property

$$\begin{aligned} \delta(\omega) &= \infty \qquad \omega = 0 \\ \delta(\omega) &= 0 \qquad \text{otherwise} \end{aligned} \tag{29}$$

The delta function has a sifting property

$$\int_{-\infty}^{\infty} d\omega\, f(\omega)\delta(\omega - \omega_0) = f(\omega_0) \tag{30}$$

That allows it to 'pick out' a certain value form the function with which it is convolved.
The Dirac delta function can be seen as the limit of a series of progressively more 'spiky' test functions.

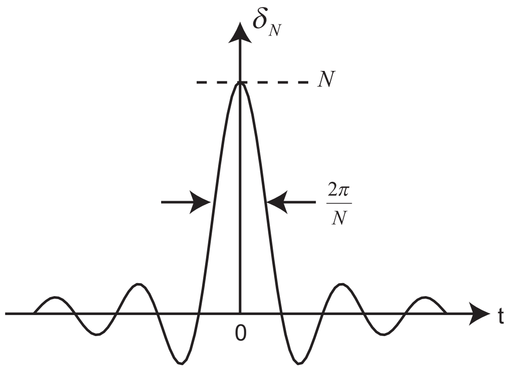
$$\delta(\omega) = \lim_{N\to\infty} \int_{-N}^{N} \frac{dt}{2\pi} e^{-i\omega t} = \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{-i\omega t} \tag{31}$$

or alternatively:

$$\delta(t) = \lim_{N\to\infty} \delta_N(t); \lim_{N\to\infty} N e^{-N^2\pi^2 t^2}; \quad \lim_{N\to\infty} N\mathrm{rect}(Nt); \quad \lim_{N\to\infty} N\frac{\sin(Nt)}{Nt} \tag{32}$$

For example: the function

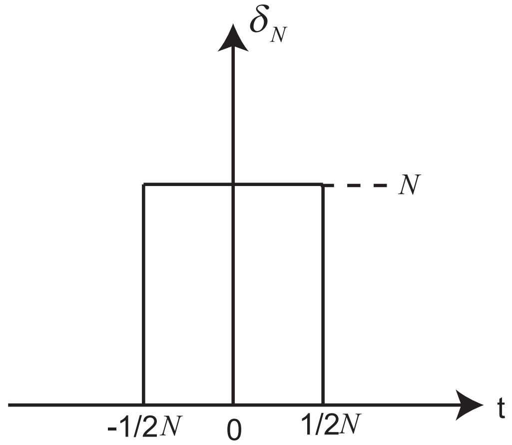
$$\delta_N(t) = \frac{\sin(Nt)}{t} \tag{33}$$



has the desired sifting and normalization properties, as does the function

$$\delta_N(t) = N\,\mathrm{rect}(Nt) \tag{34}$$

The Fourier transform of the rect(...) function is the sinc(...) function; so these two sequences are



Fourier Transform pairs. This can be seen formally from the following calculation;

$$
\begin{aligned}
\tilde{\delta}_N(\omega) &= \int_{-\infty}^{\infty} dt\, \delta_N(t) e^{i\omega t} \\
&= N \int_{-1/2N}^{1/2N} dt\, e^{i\omega t} \\
&= N \frac{e^{i\omega t}}{i\omega}\Big|_{-1/2N}^{1/2N} \\
&= N \frac{e^{i\omega/2N} - e^{-i\omega/2N}}{i\omega} \\
&= \frac{e^{i\omega/2N} - e^{-i\omega/2N}}{2i\omega/2N} \\
&= \frac{\sin(\omega/2N)}{\omega/2N} \\
&= \mathrm{sinc}(\omega/2N).
\end{aligned}
$$

5 Sampling Theorem

Often in the laboratory we can not measure a continuous variable; we 'sample' it using a discrete detector at a set of specified positions, for example. Thus if the field is represented by the analytic signal $U(\vec{r}, t)$, we usually end up with a set of numbers representing the signal, say $\{U_n\}$, where

$$
U_n = U(\vec{r}_n, t_n) \qquad \underbrace{\delta\vec{r}\ \delta t}_{\text{sampling volume}} \tag{35}
$$

Under what conditions can we say that the sample set is a faithful representation of the signal itself? We can use the Fourier series to answer this question. First we need an intermediate result.
Let $f(x)$ be an aperiodic integrable function. Then $g(x) = \sum_{n=-\infty}^{\infty} f(x+n)$ is a periodic function (since $g(x+m) = g(x)$, with m integer). Therefore $g(x)$ must have a Fourier series representation.

$$
g(x) = \sum_{n=-\infty}^{\infty} f(x+n) = \sum_{m=-\infty}^{\infty} \tilde{f}_m e^{imx} \tag{36}
$$

where

$$
\tilde{f}_m = \int_0^1 dx\, g(x) e^{imx} = \int_0^1 dx \sum_{n=-\infty}^{\infty} f(x+n) e^{imx} \tag{37}
$$

The sum and integral can be combined; $\int_0^1 dx \sum_{n=-\infty}^{\infty} \cdots \rightarrow \int_{-\infty}^{\infty} dx \cdots$

$$\tilde{f}_m = \int_{-\infty}^{\infty} dx\, f(x+n) e^{imx} \tag{38}$$

Letting $x' = x + n$

$$\tilde{f}_m = \int_{-\infty}^{\infty} dx'\, f(x') e^{im(x'-n)} = e^{-imn} \tilde{f}(m) = \tilde{f}(m) \tag{39}$$

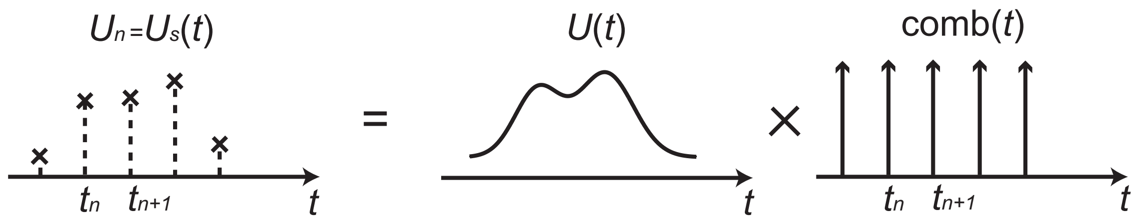
So that

$$g(x) = \sum_{n=-\infty}^{\infty} f(x+n) = \sum_{m=-\infty}^{\infty} \tilde{f}_m e^{imx} \tag{40}$$

Now consider the special case $f(x) = \delta(x)$. Substituting this in Eqns.(39) and (40), we find the Poisson Sum Formula:

$$\sum_{n=-\infty}^{\infty} \delta(x+n) = \sum_{m=-\infty}^{\infty} e^{imx}. \tag{41}$$

This is very useful for representing sampled functions, because we can think of a sampled function as being a product of the continuous function with a 'comb' of Dirac delta functions.



Then if $U_s(t)$ represents the sampled function, so $U_n = U_s(t_n)\Delta t$;

$$U_s(t) = U(t) \times \sum_{n=-\infty}^{\infty} \delta(t - n\tau_s) \tag{42}$$

where τ_s is called the sampling rate. The spectrum of this function is the convolution

$$\tilde{U}_s(\omega) = \tilde{U}(\omega) * \mathcal{F}\left\{ \sum_{n=-\infty}^{\infty} \delta(t - n\tau_s) \right\} \tag{43}$$

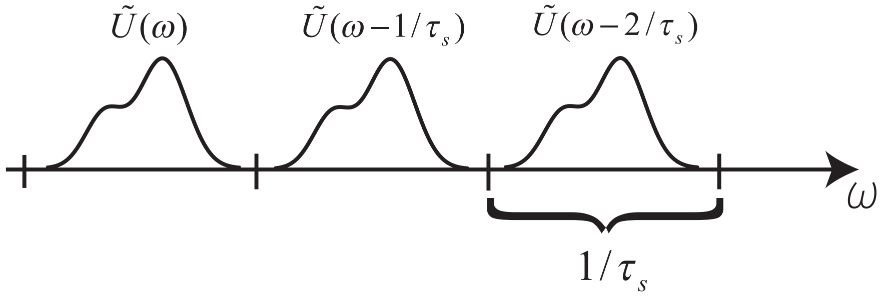
and the Fourier transform of the second term is

$$\mathcal{F}\left\{ \sum_{n=-\infty}^{\infty} \delta(t - n\tau_s) \right\} = \int_{-\infty}^{\infty} dt \sum_{n=-\infty}^{\infty} \delta(t - n\tau_s) e^{i\omega t} = \int_{-\infty}^{\infty} dt \sum_{n=-\infty}^{\infty} e^{int/\tau_s - i\omega t} \tag{44}$$

where in the last step we used the Poisson Sum Formula. Therefore

$$\tilde{U}_s(\omega) = \tilde{U}(\omega) * \sum_{n=-\infty}^{\infty} \delta(\omega - n/\tau_s) \tag{45}$$

is a periodic function, with the spectrum of the signal occuring in every period, as shown in the figure. It is clear that we can get back the orginal function $U(t)$ by simply filtering one of these 'replica' spectra and taking its inverse transform. But this will only work if the replicas do not overlap. The condition for 'non-overlapping' is that $1/\tau_s$ is broader in frequency than the spectrum of the pulse itself. This leads to the Nyquist sampling theorem:
To faithfully reconstruct a pulsed signal of bandwidth (FWHM) $\Delta\omega$, the sampling rate in the time domain must be greater than $\tau_s = 2\pi/\Delta\omega$.



$\tilde{U}(\omega)$
$\tilde{U}(\omega - 1/\tau_s)$
$\tilde{U}(\omega - 2/\tau_s)$
ω
$1/\tau_s$