Optical Field Mixing

Second harmonic generation

$$E = E_0 \cos \omega t$$

$$P(E) = \varepsilon_0 \chi^{(2)} E_0^2 \cos^2 \omega t = \frac{1}{2} \varepsilon_0 \chi^{(2)} E_0^2 (\cos 2\omega t + 1)$$

Sum and difference generation

$$E = E_1 \cos \omega_1 t + E_2 \cos \omega_2 t$$

$$P(E) = \varepsilon_0 \chi^{(2)} (E_1 \cos \omega_1 t + E_2 \cos \omega_2 t)^2$$

$$\Rightarrow 2\varepsilon_0 \chi^{(2)} E_1 E_2 \cos \omega_1 t \cos \omega_2 t$$

$$= \varepsilon_0 \chi^{(2)} \left[\cos(\omega_1 + \omega_2) t + \cos(\omega_1 - \omega_2) t \right]$$

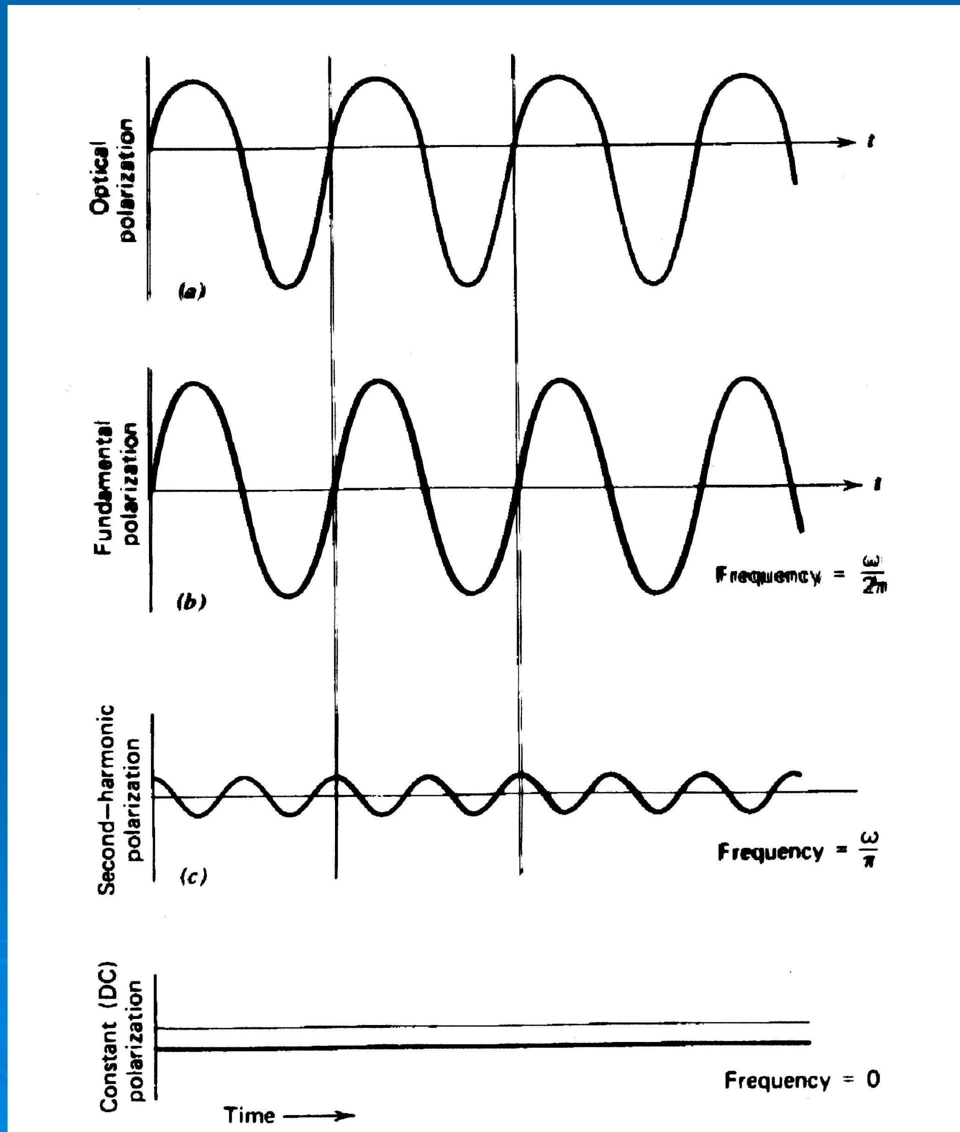
Oscillating Polarisation

Optical polarisation

Fundamental polarisation

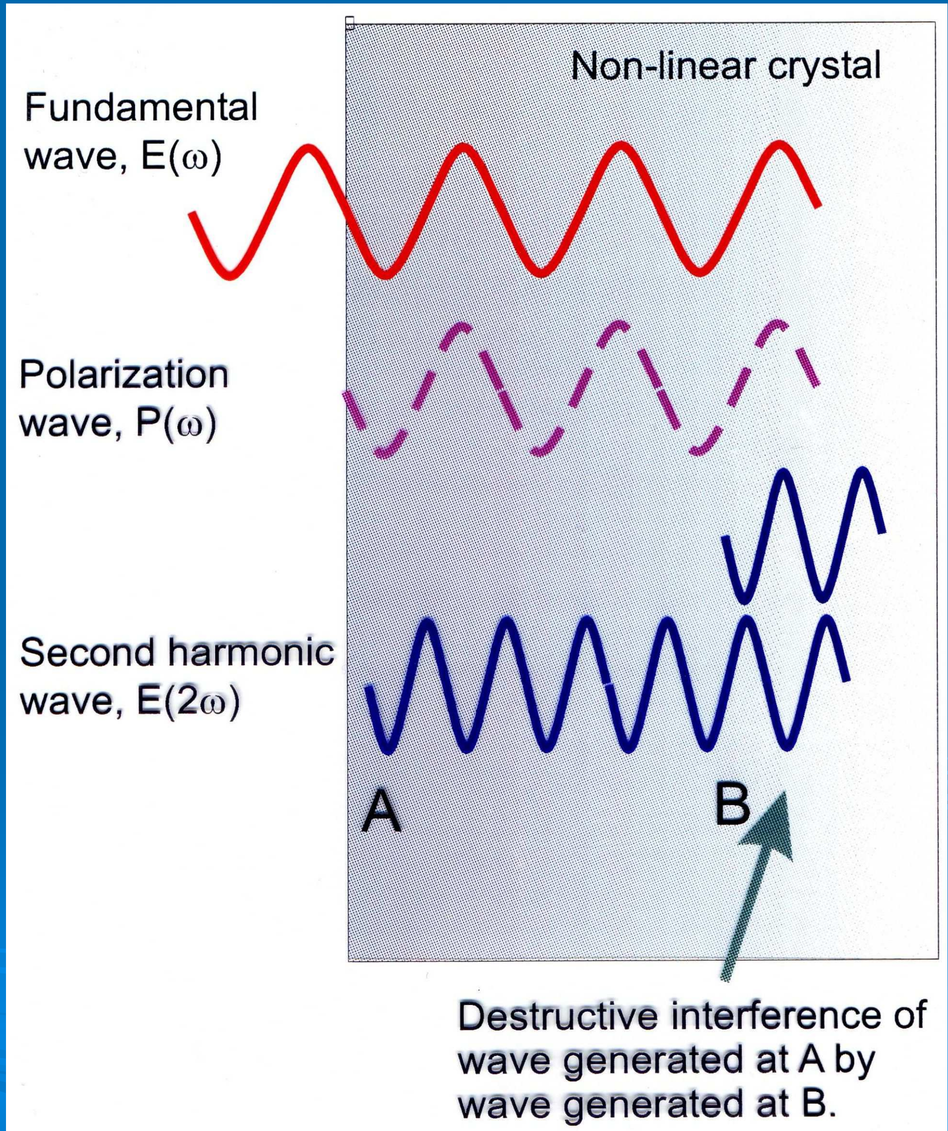
SH Polarisation

Constant (dc) polarisation

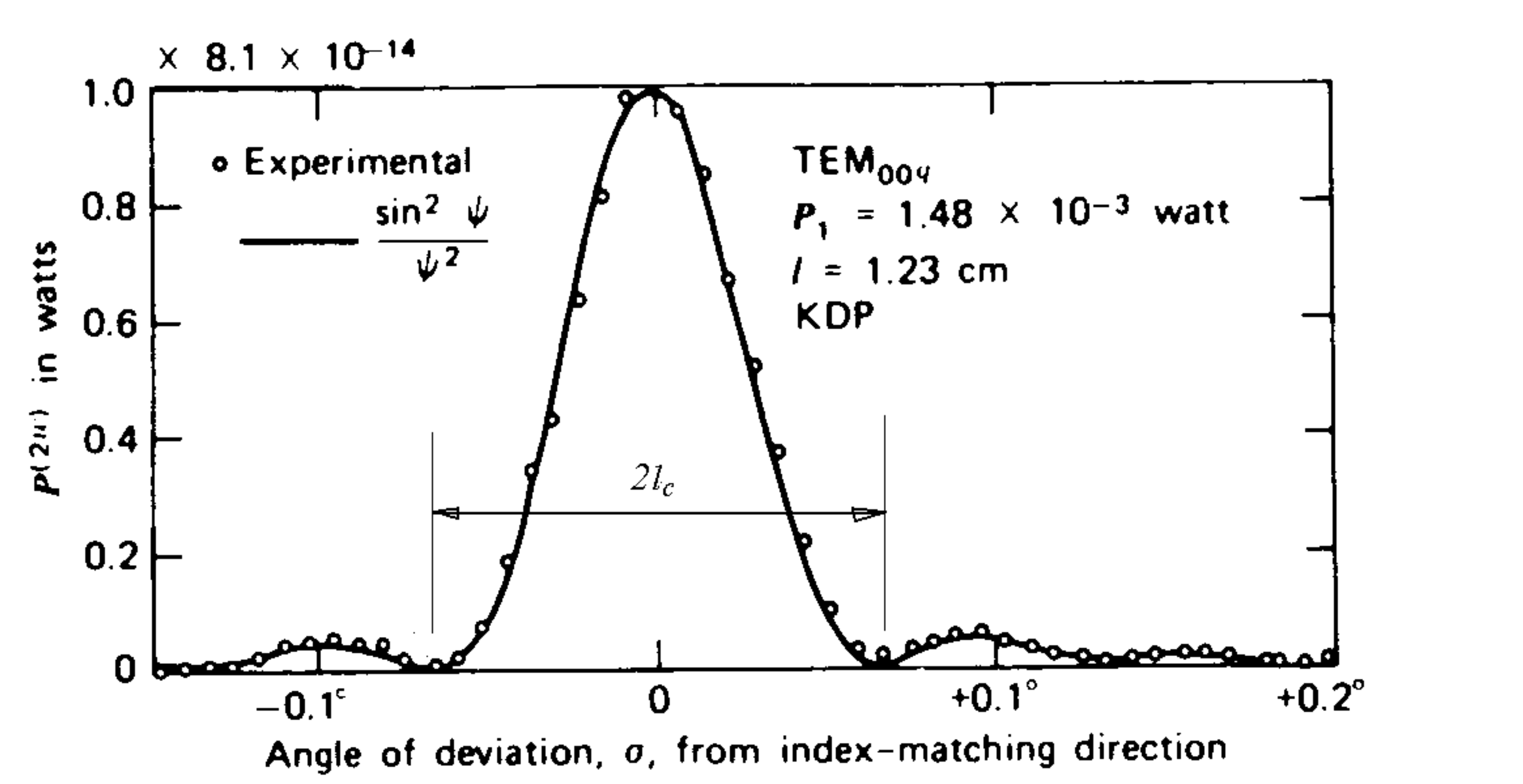


SHG: Efficiency factors

The generated second harmonic has to remain “in step” with the fundamental wave which produces it. This is known as *phase-matching.*



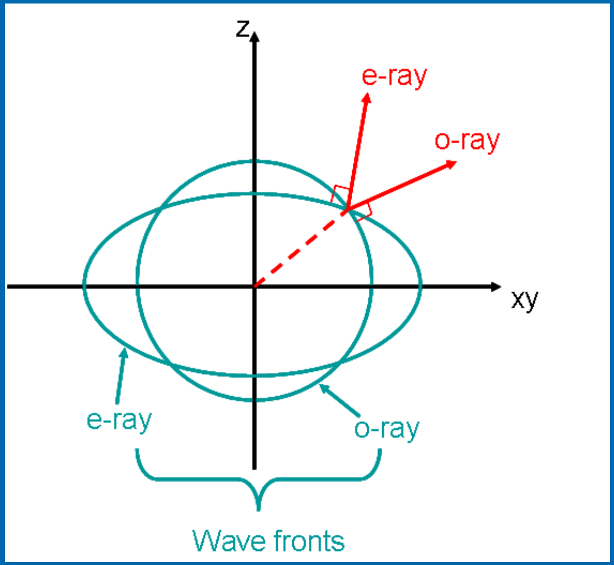
Coherence length

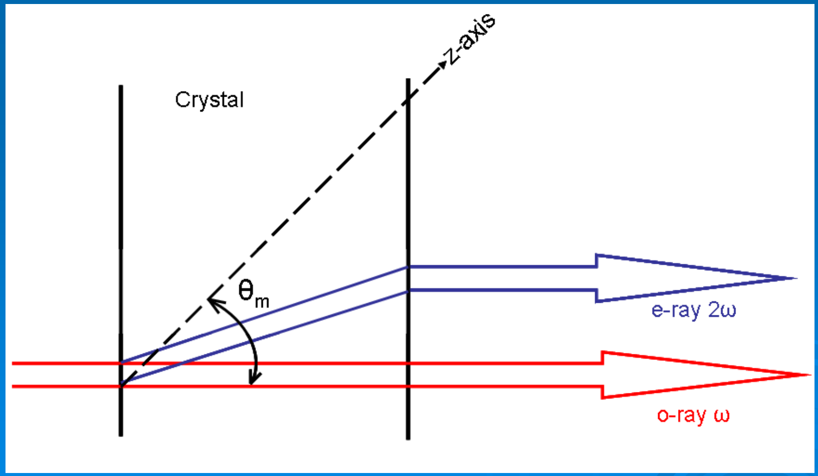


$$I^{2\omega} = \frac{(2\omega)^2(\frac{1}{2}\chi^{SHG})^2}{2n^{2\omega}(n^{\omega})^2c^3\varepsilon_0}(I^{\omega})^2\left\{\frac{\sin(\Delta kz/2)}{\Delta kz/2}\right\}^2 z^2$$

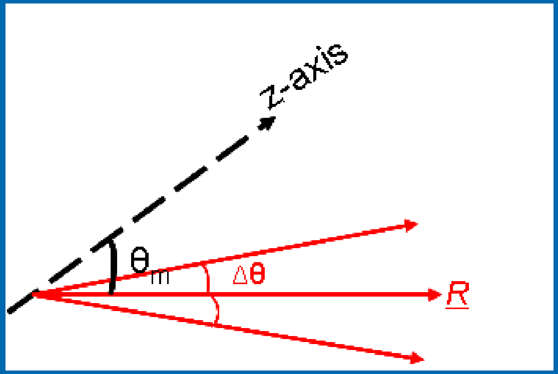
Efficiency Factors

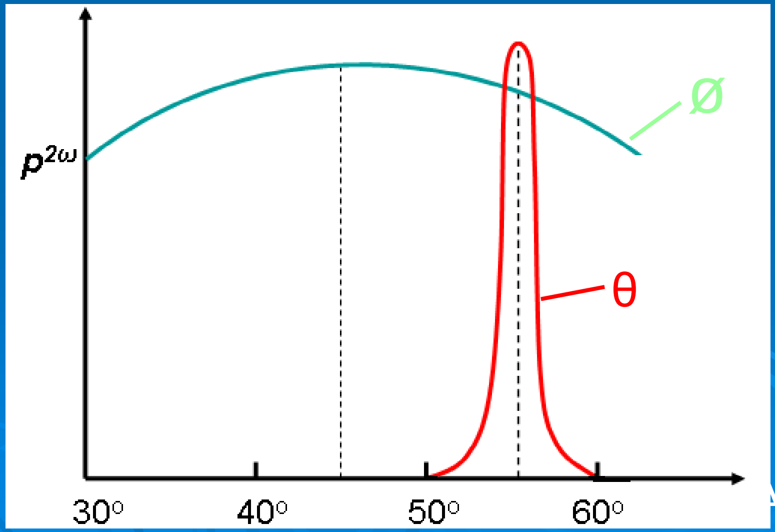
1. Walk-off





2. Beam Divergence

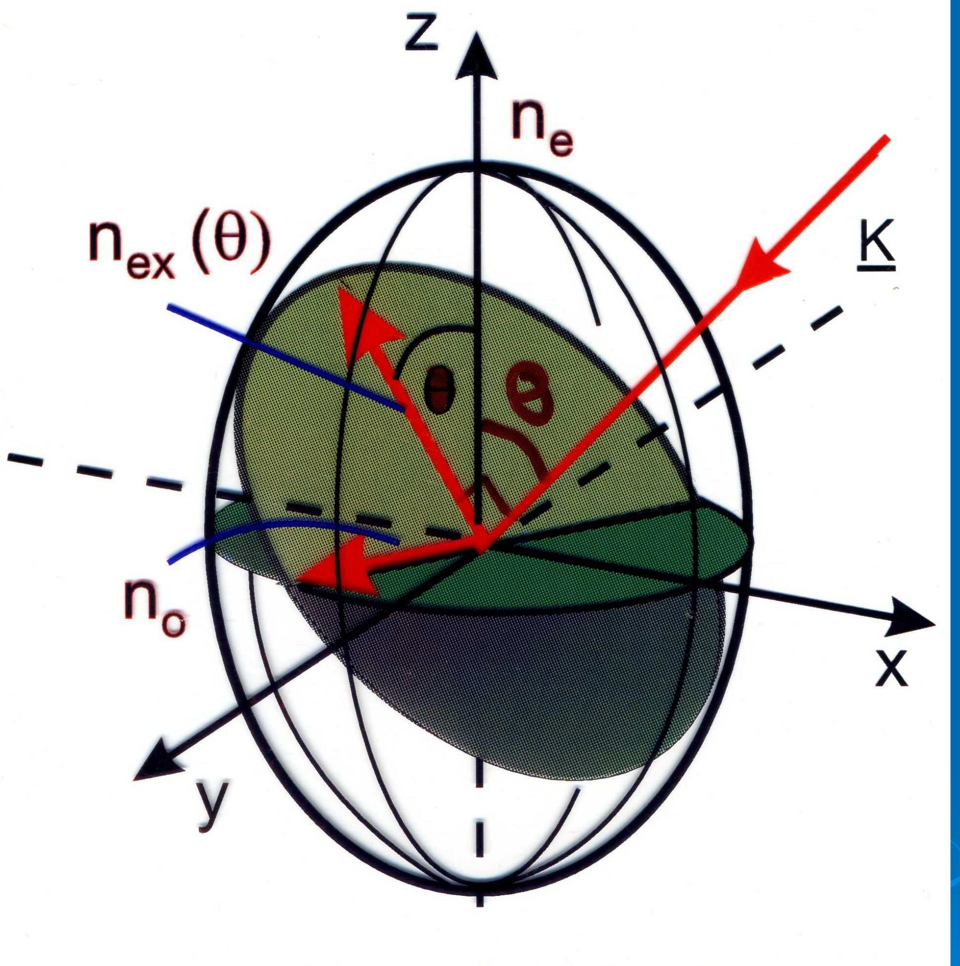




Angle

Index ellipsoid

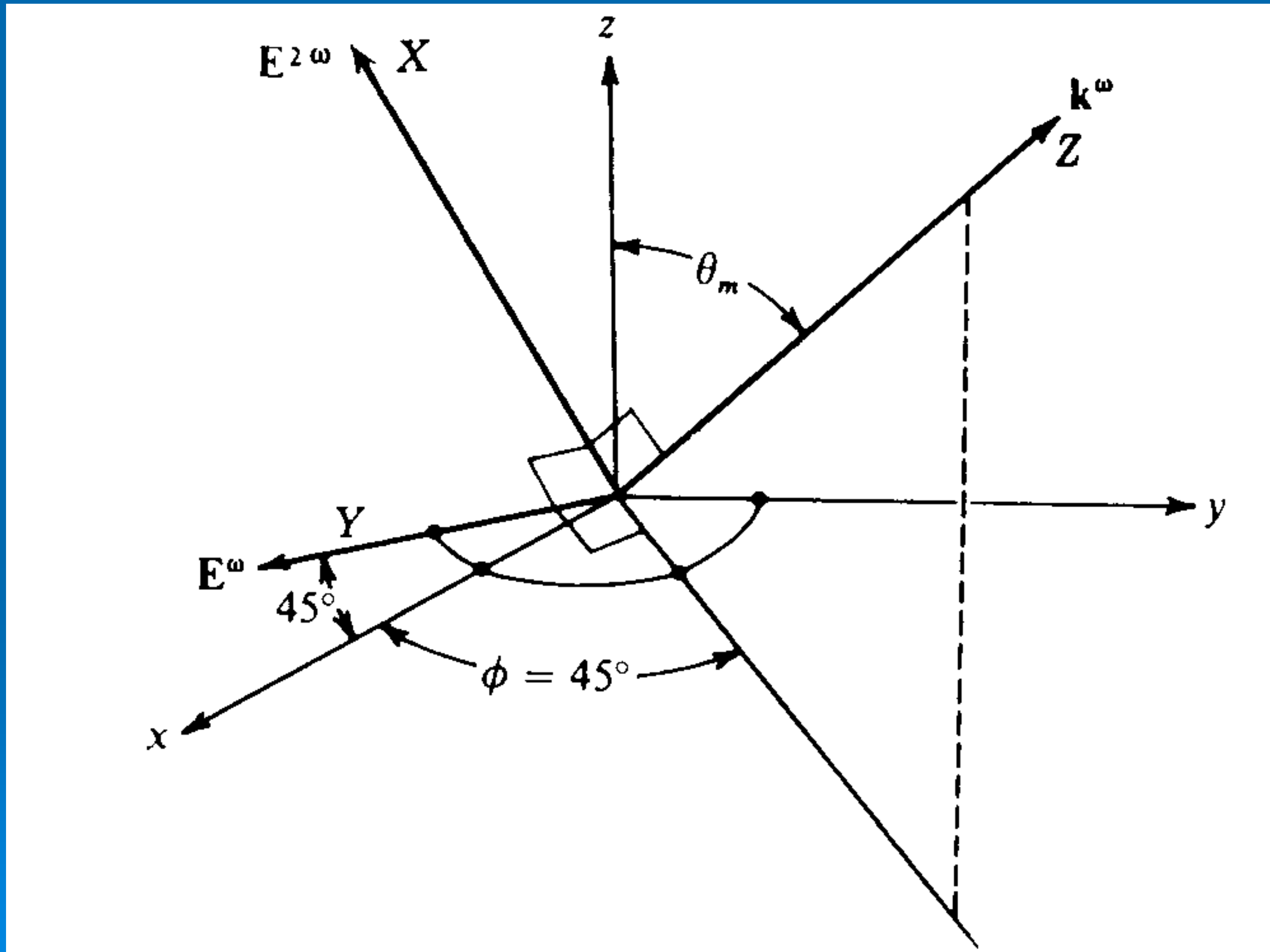
To match the indices the fundamental wave propagates as the ordinary wave while the second harmonic as the extraordinary.



Index ellipsoid for uniaxial crystal

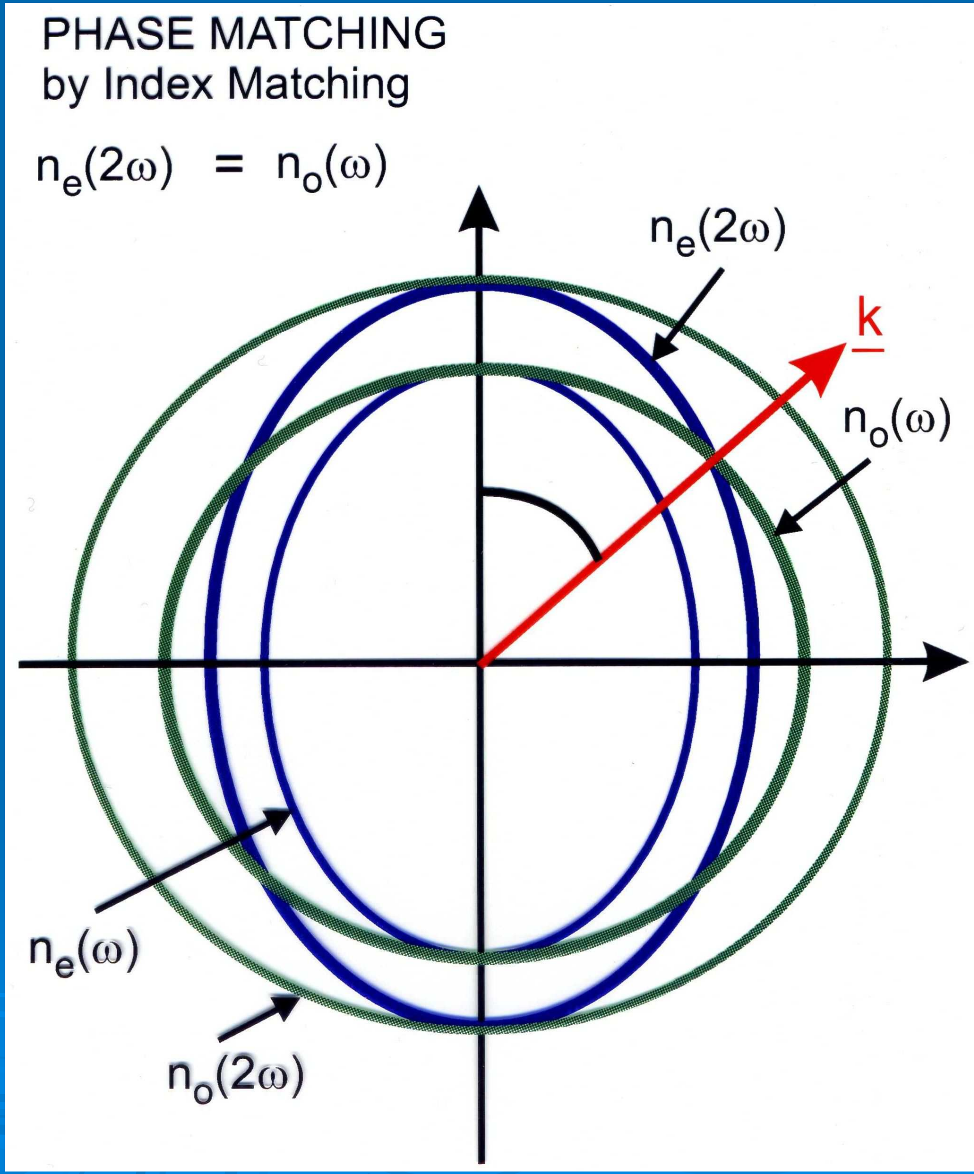
$$\frac{x^2}{n_o^2}+\frac{y^2}{n_o^2}+\frac{z^2}{n_e^2} = 1$$

Directions and fields in SHG



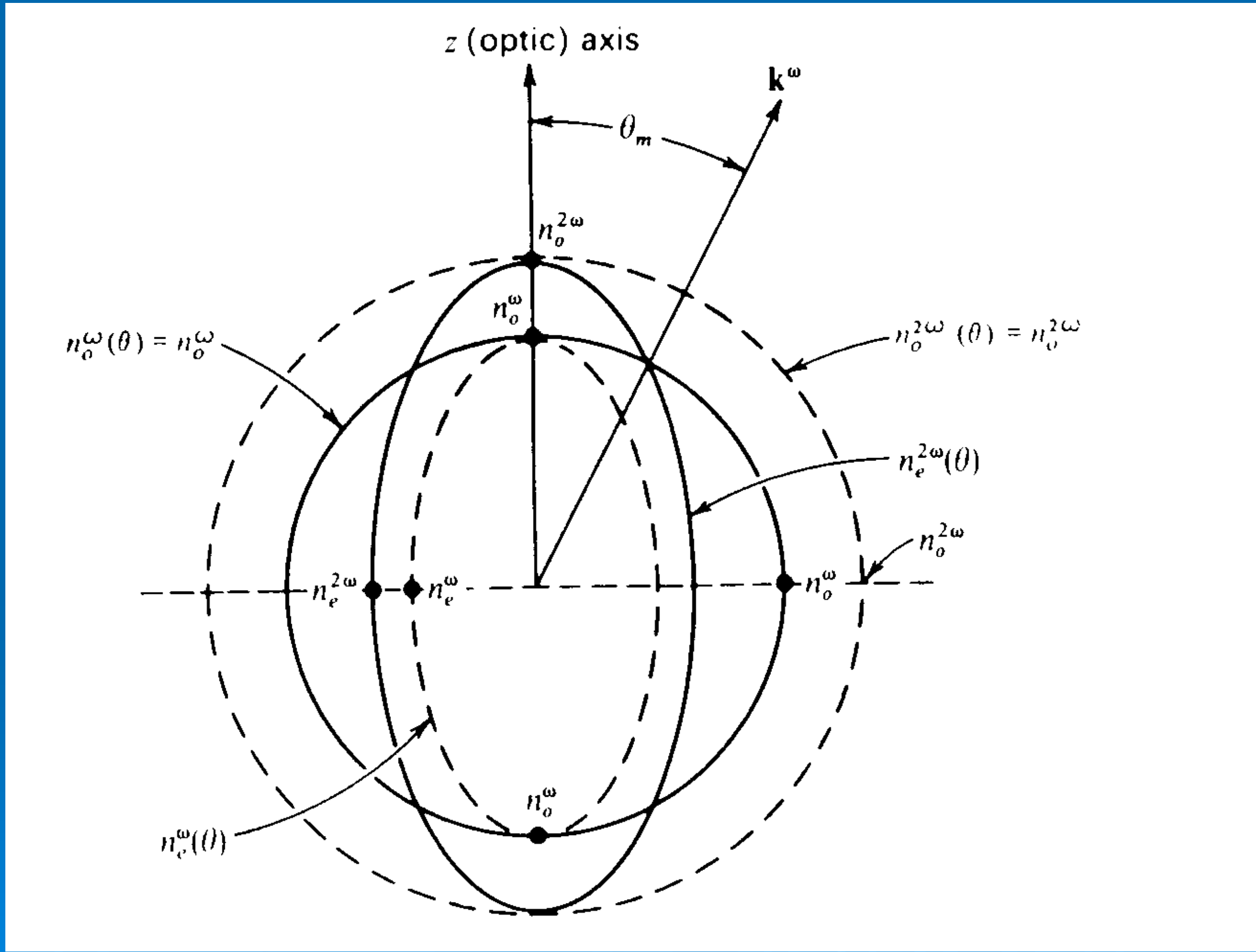
Phase-matching

The idea is to make use of *dispersion* to achieve a common index of refraction and thereby a common propagation velocity in the medium.



Phase-matching - Index ellipsoid

In type I phase-matching the fundamental and second harmonic waves travel as waves of different types,
i.e. one as the ordinary the other as the extraordinary wave.



The required condition is then $n1(2\omega)=n2(\omega)$

Computing the phase-match angle

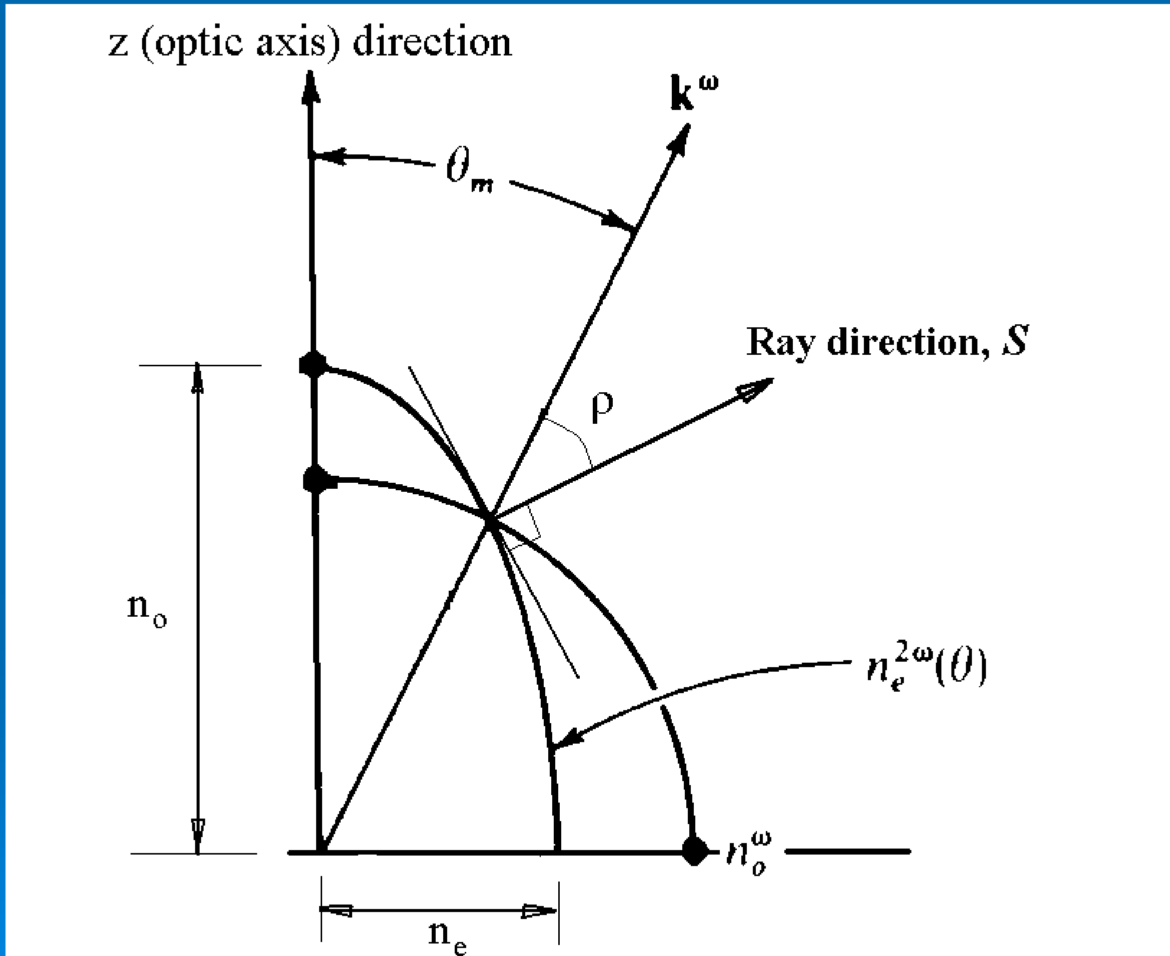
For the extraordinary wave the index ellipse gives:

$$\frac{1}{n_e^2(\theta)} = \frac{\cos^2\theta}{n_o^2} + \frac{\sin^2\theta}{n_e^2}$$

So, if we require $n_e\,(2\omega,\theta) = n_o(\omega)$ then,

$$\theta_m = \cos^{-1}\left\{\frac{\left(n_0^{\omega}\right)^{-2} - \left(n_e^{2\omega}\right)^{-2}}{\left(n_0^{2\omega}\right)^{-2} - \left(n_e^{2\omega}\right)^{-2}}\right\}^{\frac{1}{2}}$$

Walk-off



The second harmonic wave can walk away from the fundamental.

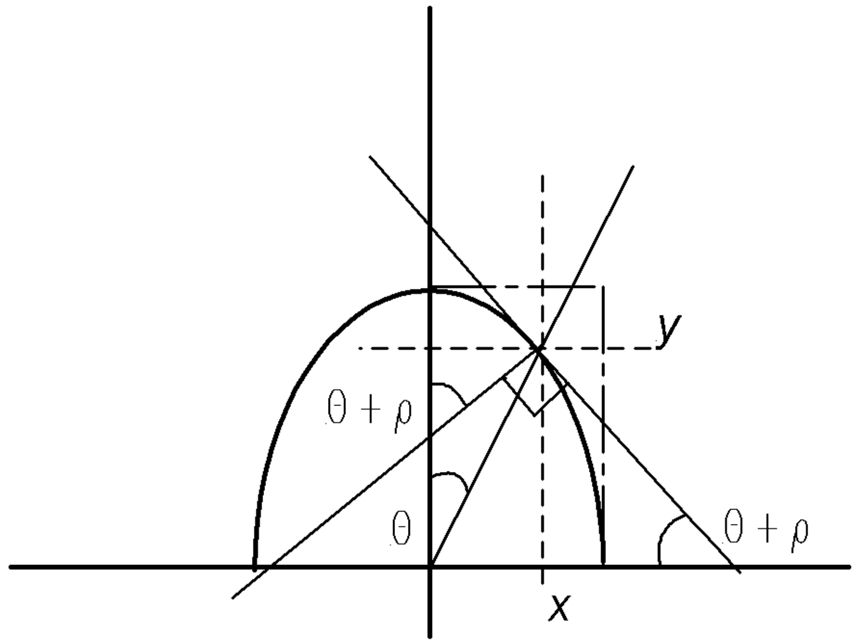
S and *k* are not collinear.

Walk-off calculation

$$\tan(\theta+\rho) = \left(\frac{dy}{dx}\right) = -\left(\frac{x}{y}\right)\left(\frac{b^2}{a^2}\right) = -\tan\theta \times \left(\frac{b^2}{a^2}\right)$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

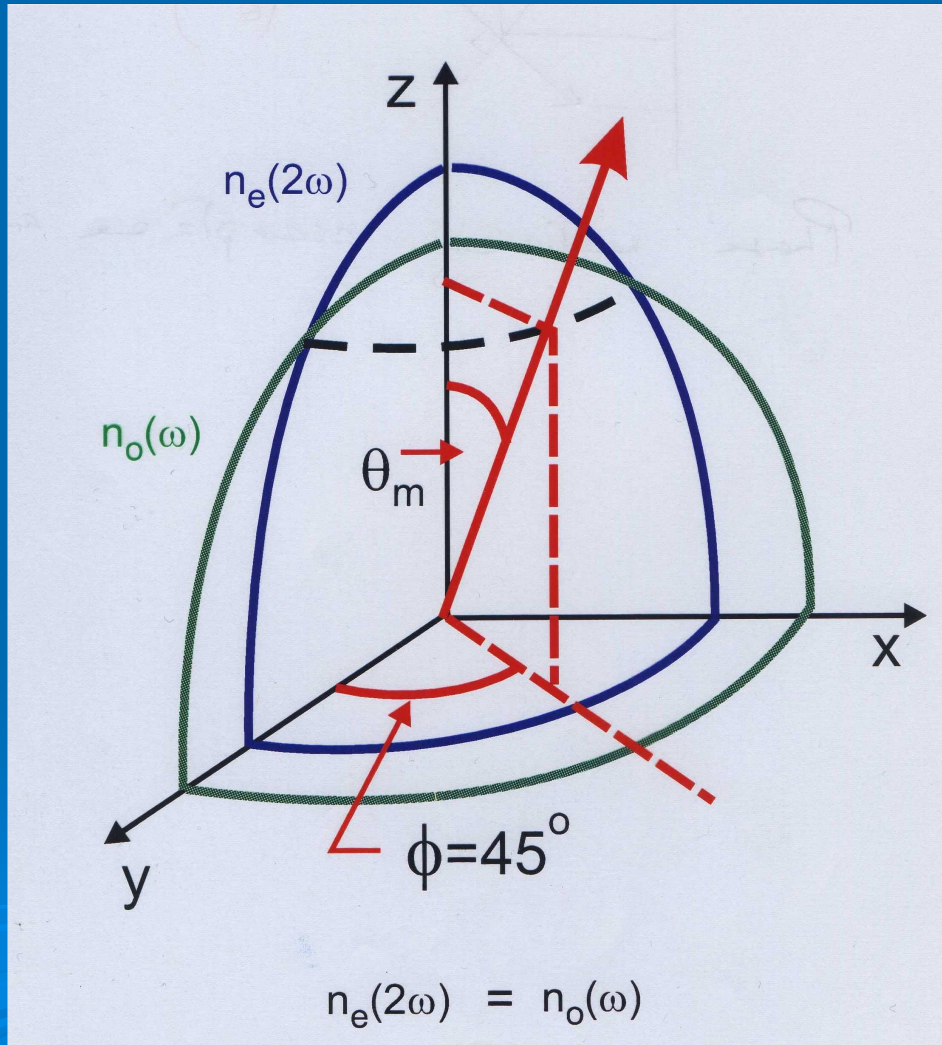


Maximising the SHG output

- When the angle θ to z-axis is equal to θ_m
- When the angle to the x-axis equals 45°
- When the input beam has a low divergence
- When the crystal temperature is constant (since $n = f(T)$)
- When the crystal is relatively short (coherence length)

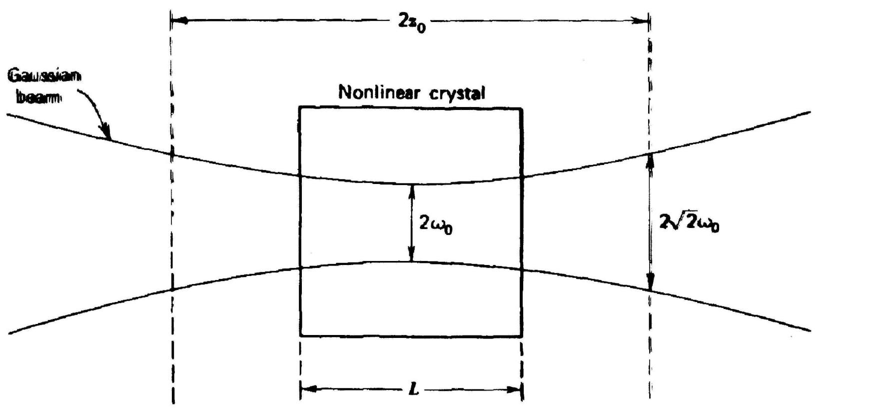
45 degree, z-cut

- Efficiency maximised
- Little or no walk-off
- High angular tolerance

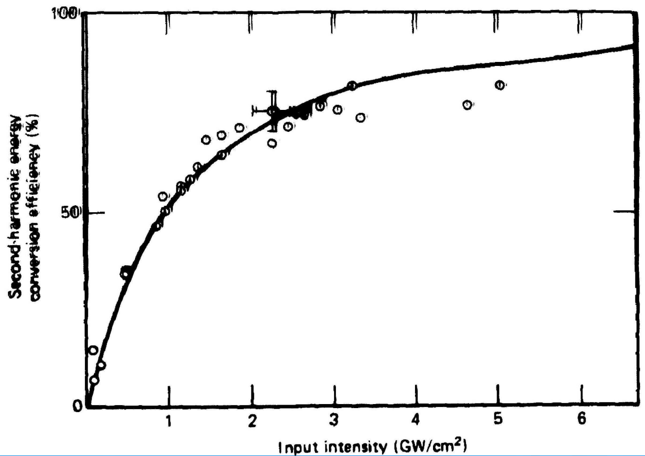


Depletion & Focus

Coupled Equations now give a *tanh* solution



$$z_0 = \frac{\pi w_0^2 n}{\lambda}$$



$$\frac{I^{2\omega}}{I^{\omega}} = \tanh^2\left(\frac{\kappa L}{2}\right)$$

Depleted Input Beam

For depletion under perfect phase-matching:

$$\frac{I^{2\omega}}{I^{\omega}} = \tanh^2[\frac{\kappa L}{2}]$$

Where the coupling is given by:

$$\kappa^2 = 8d^2 \left(\frac{\mu_0}{\varepsilon_0}\right)^{3/2} \frac{\omega^2 I^{\omega}(0)}{n^{2\omega}(n^{\omega})^2}$$

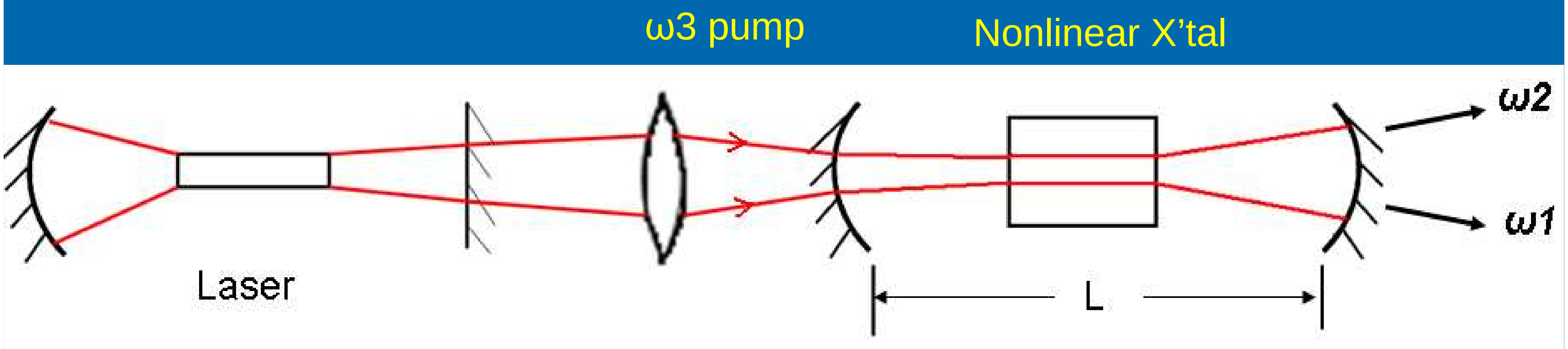
Which for low depletion reduces to:

$$\frac{I^{2\omega}}{I^{\omega}} = 2d^2 \left(\frac{\mu_0}{\varepsilon_0}\right)^{3/2} \frac{\omega^2 I^{\omega}(0) L^2}{n^{2\omega}(n^{\omega})^2}$$

Which is identical to our earlier expression, noting

$$d = \frac{1}{2}\varepsilon_0 \chi^{SHG}$$

Optical Parametric Oscillator

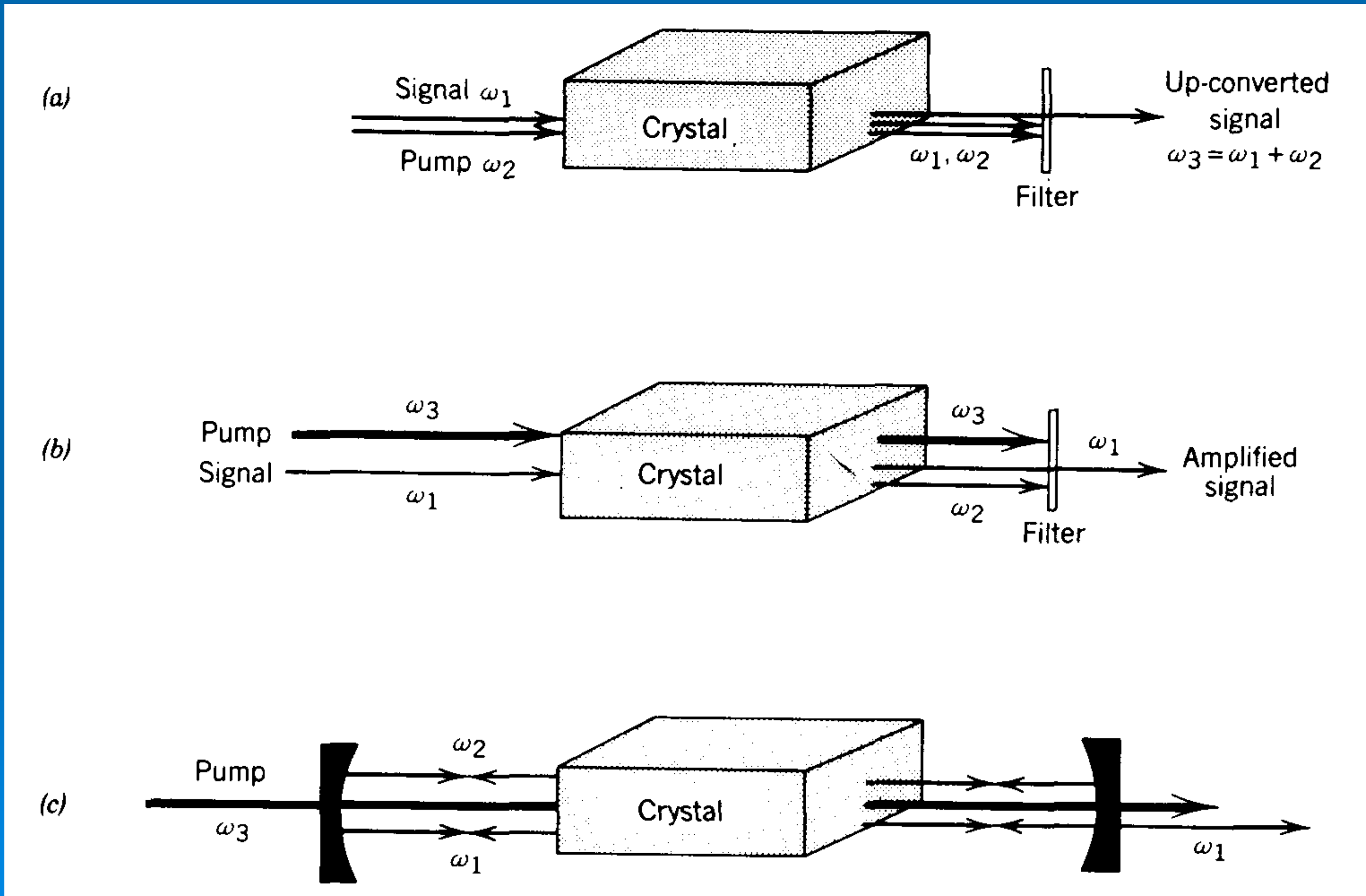


Conditions for the generation of new optical frequencies:

$$\omega_1 + \omega_2 = \omega_3 \quad Energy$$
$$k_1 + k_2 = k_3 \quad Momentum$$

$$\frac{\omega_1 n_1 L}{c} = m\pi \quad Modes$$
$$\frac{\omega_2 n_2 L}{c} = s\pi \quad Modes$$

Parametric processes I



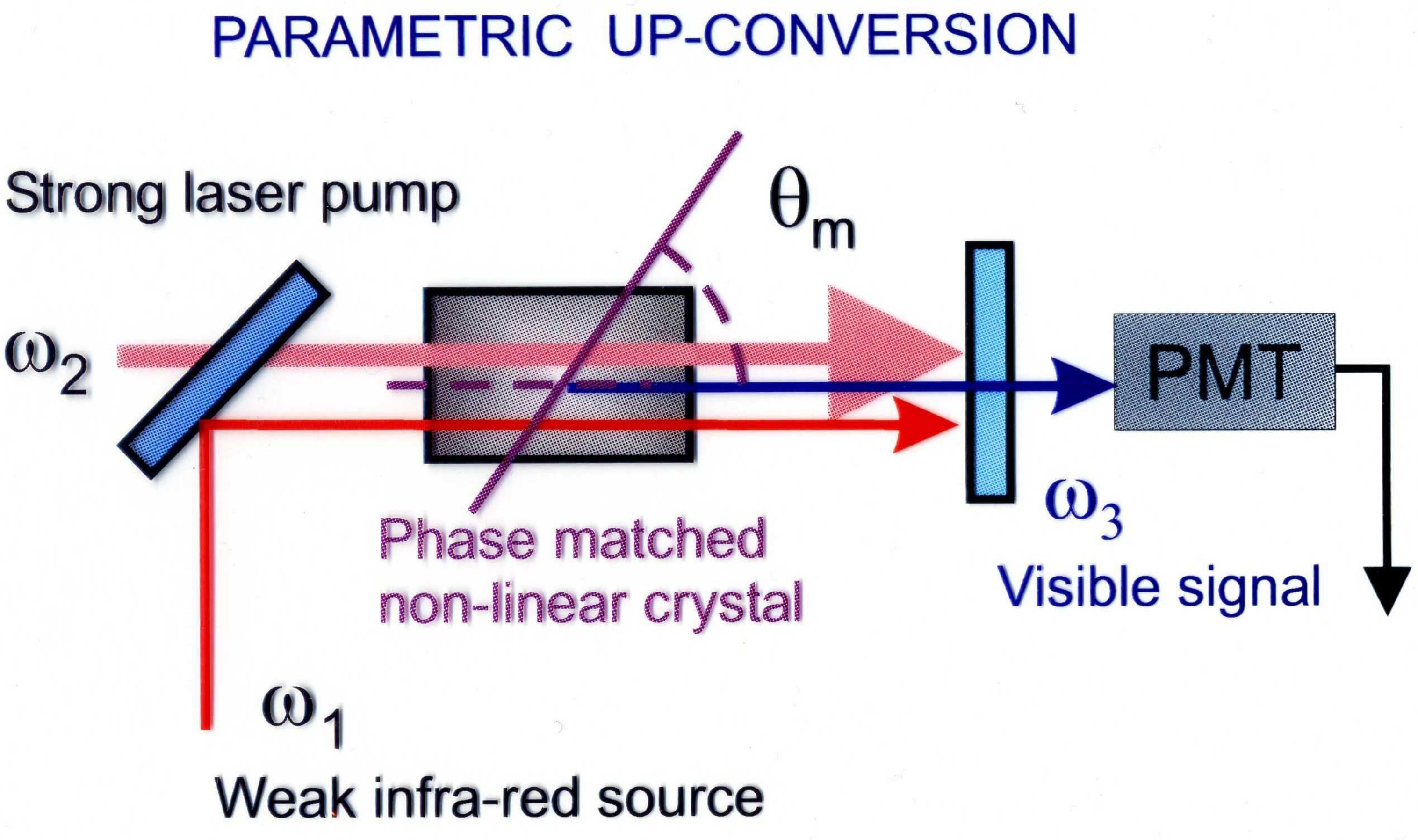
Parametric processes II

$$E(t) = \mathrm{Re}\{E(\omega_1)\exp(-i\omega_1 t) + E(\omega_2)\exp(-i\omega_2 t)\}$$

$$P_{NL} = 2dE(t)^2$$

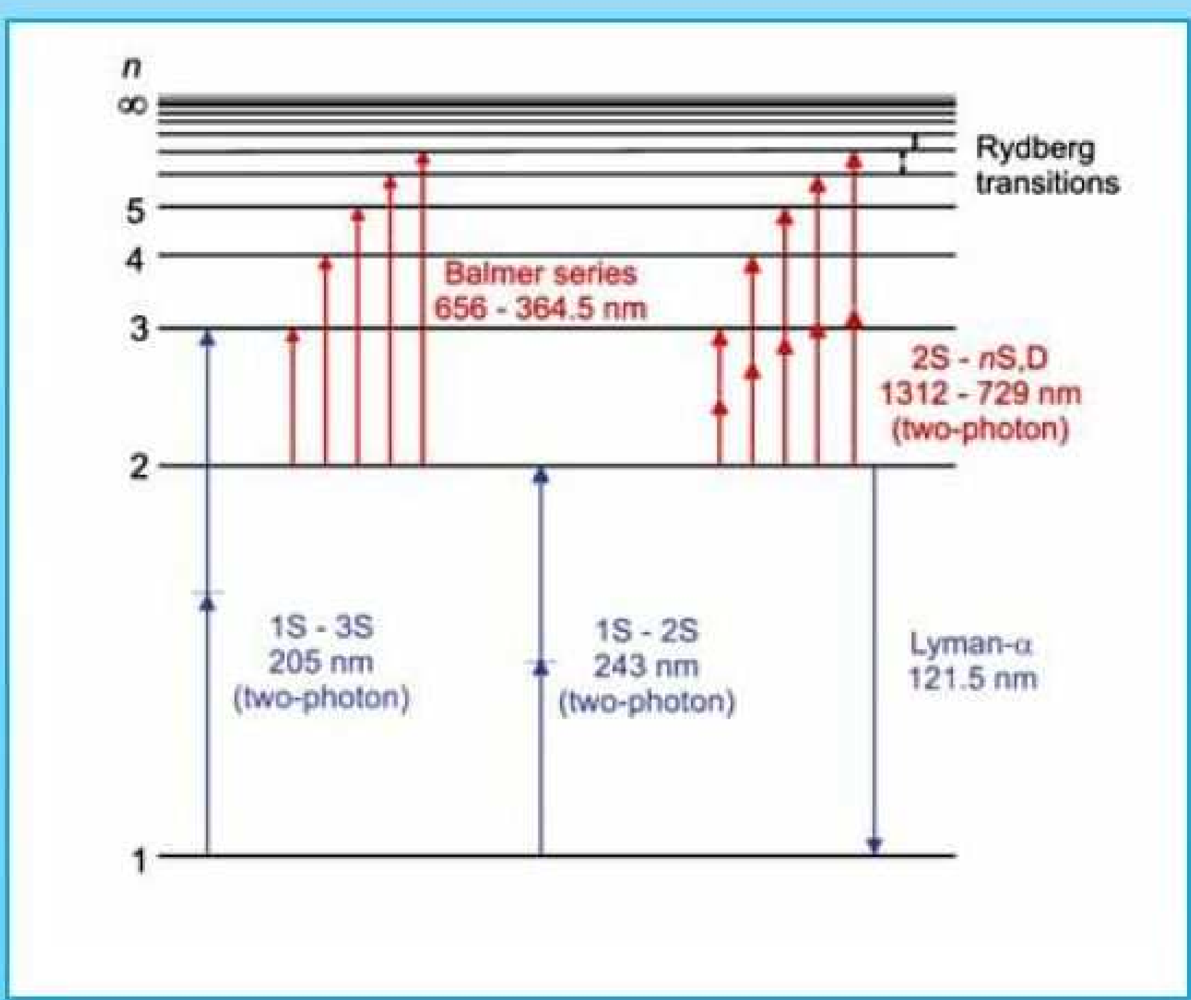
$$\begin{aligned}
P_{NL}(0) &= d\left[|E(\omega_1)|^2 + |E(\omega_2)|^2\right] \\
P_{NL}(2\omega_1) &= d\,E(\omega_1)E(\omega_1) \\
P_{NL}(2\omega_2) &= d\,E(\omega_2)E(\omega_2) \\
P_{NL}(\omega_+) &= 2d\,E(\omega_1)E(\omega_2) \\
P_{NL}(\omega_-) &= 2d\,E(\omega_1)E^*(\omega_2)
\end{aligned}$$

Parametric Processes III



$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 \qquad \omega_3 = \omega_1 + \omega_2$$

Example: generation of 243nm



Two-photon transitions in hydrogen

Example: summing in KDP

$$\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \varepsilon_0 \begin{bmatrix} 0 & 0 & 0 & d_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & d_{25} & 0 \\ 0 & 0 & 0 & 0 & 0 & d_{36} \end{bmatrix} \begin{bmatrix} E_x^2 \\ E_y^2 \\ E_z^2 \\ 2E_yE_z \\ 2E_xE_z \\ 2E_xE_y \end{bmatrix}$$

For this symmetry group: $d_{14} = d_{25} \neq d_{36}$

For the second harmonic to propagate as the extraordinary wave: $P_i^{2\omega} = 2d_{ijk}E_j^{\omega}E_k^{\omega}$

Symmetry & Kleinman

Since no physical significance is attached to an exchange of E_j and E_k

$$d_{ijk} = d_{ikj}$$

And using the contracted form

$$xx = 1;\ yy = 2;\ zz = 3$$

$$yz = zy = 4;\ xz = zx = 5;\ xy = yx = 6$$

$\oint d(P.E) = 0$

$\oint E.dP = 0$

$\oint p.dE = 0$

$\nabla_E \times P =$

In addition for a *lossless* medium all *d*-coefficients that are related by a re-arrangement of order of the subscripts are equal. This reduces the maximum number of *d's* from 18 to 10, *e.g.* for KDP: $d_{14}=d_{36}$

$$P_i = -\nabla U(E)$$

$$U(E) = -\varepsilon_0 \frac{\chi_{ij}}{2} E_i E_j - \frac{2d_{ijk}}{3} E_i E_j E_k \ldots$$

$$P_i = -\frac{\partial U(E)}{\partial E_i} = \varepsilon_0 \chi_{ij} E_j + 2d_{ijk} E_j E_k \ldots$$

See Yariv P384

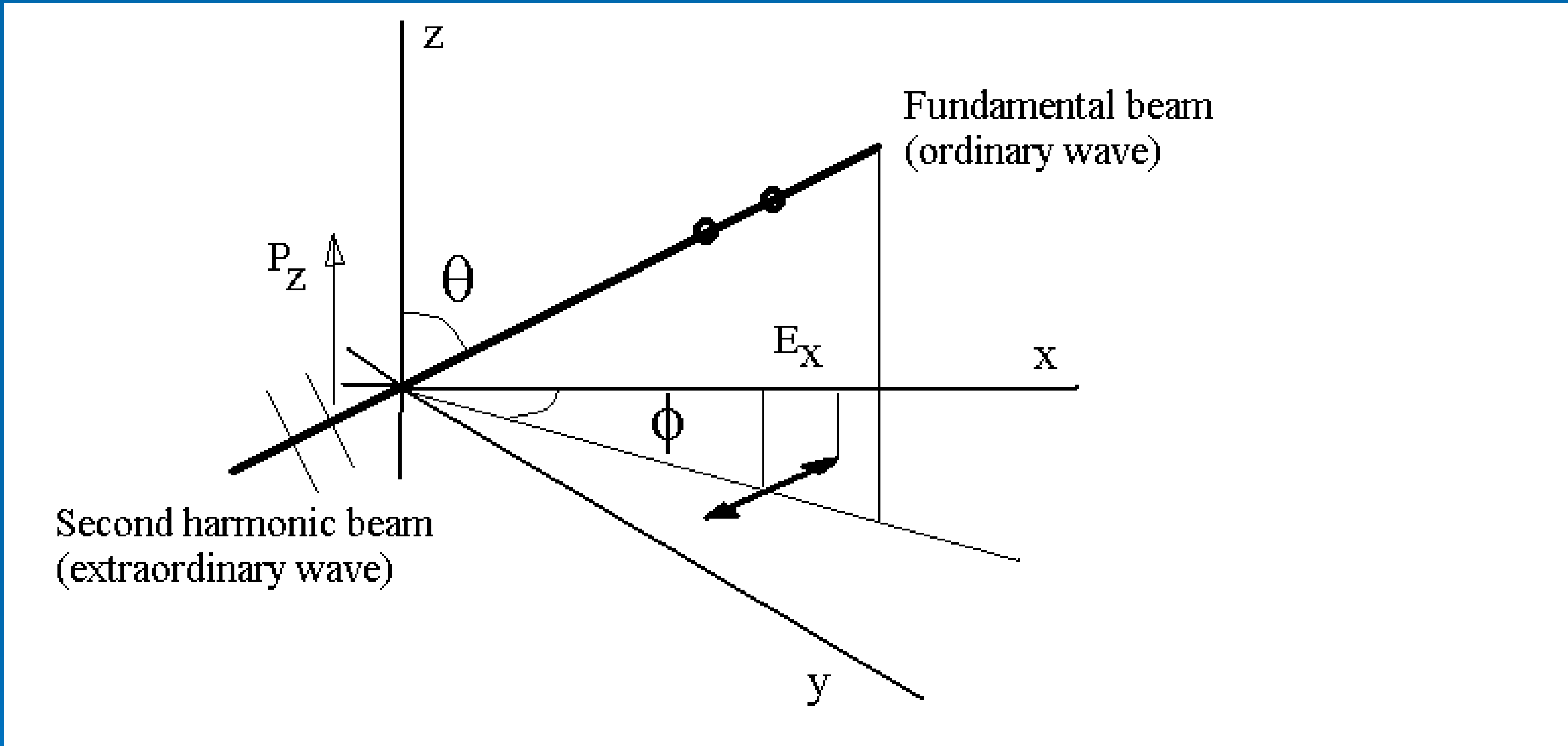
Frequency summing in KDP

The form of the *polarizability* tensor for these negative *uniaxial* crystals with 42m symmetry class enables generation of the SHG as an extraordinary wave.

The expression for d_{eff} maximises when $\theta = 90°$ and $\Phi = 45°$, *i.e.,*

$$
\begin{aligned}
P_z' &= P_z \sin\theta = 2\varepsilon_0 d_{36} E_0^2 \sin\theta(\cos\phi \times \sin\phi) \\
&= \varepsilon_0 d_{36} E_0^2 \sin\theta \sin 2\phi
\end{aligned}
$$

Phase-matching angles



The fundamental is plane-polarized in the *x-y* plane – it propagates as an *ordinary* wave and generates a SHG polarization along *z*. This has to be project along ***k***

Frequency summing in KDP

Energy conservation (ω):

$$\frac{1}{243} - \frac{1}{351} = \frac{1}{\lambda_2}$$

Indices must satisfy (k):

$$\frac{n_0}{351} + \frac{n_0'}{789} = \frac{n_e}{243}$$

Calculating the phase-matching angle gives almost 90°, i.e.,

$$\theta = \sin^{-1}\left\{\frac{n_e(\lambda_3)}{n_e^\theta(\lambda_3)}\sqrt{\frac{n_0^2(\lambda_3) - n_e^\theta(\lambda_3)^2}{n_0^2(\lambda_3) - n_e^2(\lambda_3)}}\right\}$$

$$\theta = 85.5^0$$

Temperature tuning the crystal

Since the refractive index is temperature dependent it may be possible to phase-match at 90° by exploiting this variation, *i.e.,*

$$\left(\frac{n_0}{351}+\frac{n_0'}{789}-\frac{n_e}{243}=\Delta T\times\left\{\left(\frac{dn_0/dT}{351}\right)_{351}+\left(\frac{dn_0/dT}{790}\right)_{790}-\left(\frac{dn_e/dT}{243}\right)_{243}\right\}\right)$$

Solve to find *ΔT*.

Finally, the crystal can be cut so as to have Brewster faces for the fundamental beams; the SHG is orthogonally-polarised and suffers some Fresnel loss in a single pass out of the crystal.

The final crystal design