Non-linear Optics II
(Modulators & Harmonic Generation)

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MT2010

Electro-optic modulation of light

An electro-optic crystal is essentially a variable phase plate and as such can be used either as an amplitude (intensity) modulator or as a phase modulator. One arrangement for this was shown in the last lecture. Of course the field may be applied in various directions and some examples are shown below.

We shall consider in a little detail here the longitudinal case.
The input beam is specified by

$$E_{x'} = E_{y'} = A; \;\; E_y = 0 \tag{1}$$

i.e. linearly polarised along x. The output is thus affected by the phase difference between the x' and y' directions which is induced by the electric field along z. The output field is thus,

$$E_{x'}(\ell) = A; \;\;\; E_{y'}(\ell) = A\exp(-i\phi) \tag{2}$$

The resultant complex field along y is then,

$$E_y(\ell) = \frac{A}{\sqrt{2}}\left(\exp(-i\phi) - 1\right) \tag{3}$$

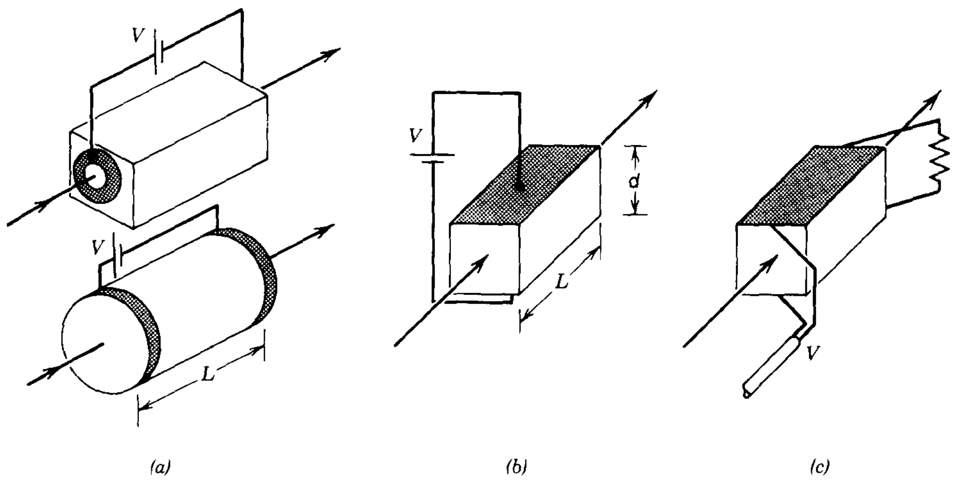


Figure 1: (a) a longitundial field, (b) a transverse field, and (c) a travelling-wave field.

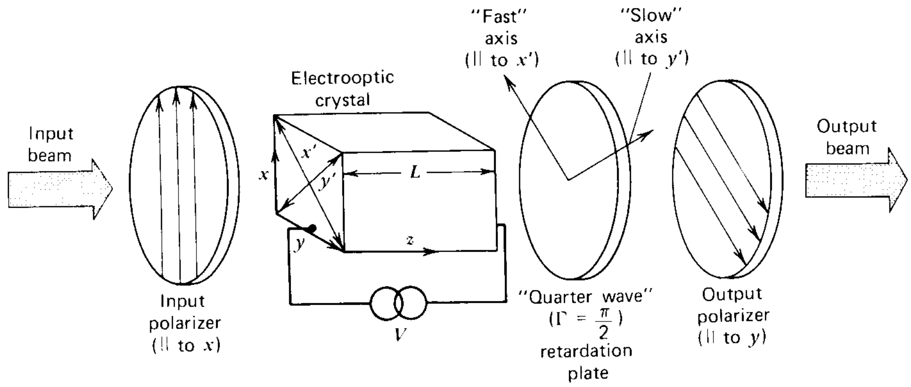


Figure 2: Longitundal modulator. The $\lambda/4$ plate provides a "bias" to 50% transmission

and the transmission ration is

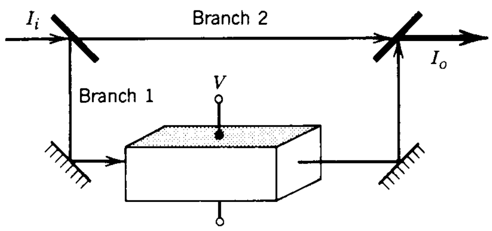
$$I_{out} = I_{in} \sin^2(\phi/2) \tag{4}$$

where

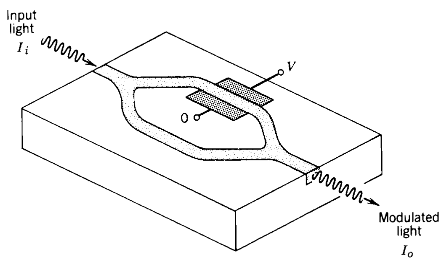
$$\phi = \frac{\pi V}{V_{\lambda/2}} \tag{5}$$

The quarter-wave plate allows modulation about the linear part of the transmission curve for fields close to zero.

A similar intensity modulator can be constructed using a Mach-Zehnder arrangement with the crystal in one of the two paths. Electro-optic modulation effectively scans the path difference of the two beam interferometer and therefore scans from say an intensity maximum to a minimum. Biasing can also be achieved using an additional $\lambda/4$ plate as above.



Principle of the Mach-Zehnder Interferometer



Optical fibre version of a Mach-Zehnder

Phase modulators.

Using now only the input polariser along say x', a varying voltage to the crystal can lead to phase and frequency modulation as follows. Let the light field be specified by

$$\begin{aligned} E_{in} &= A\cos\omega t & (6) \\ E_{out} &= A\cos(\omega t - kx + \Delta\phi) = A\cos(\omega t - \frac{\omega}{c}(n_0 - \frac{n_0^3}{2} r_{63} E_m \sin\Omega t)\ell) & (7) \end{aligned}$$

and the applied voltage by

$$V_m = V_0 \sin \Omega t \tag{8}$$

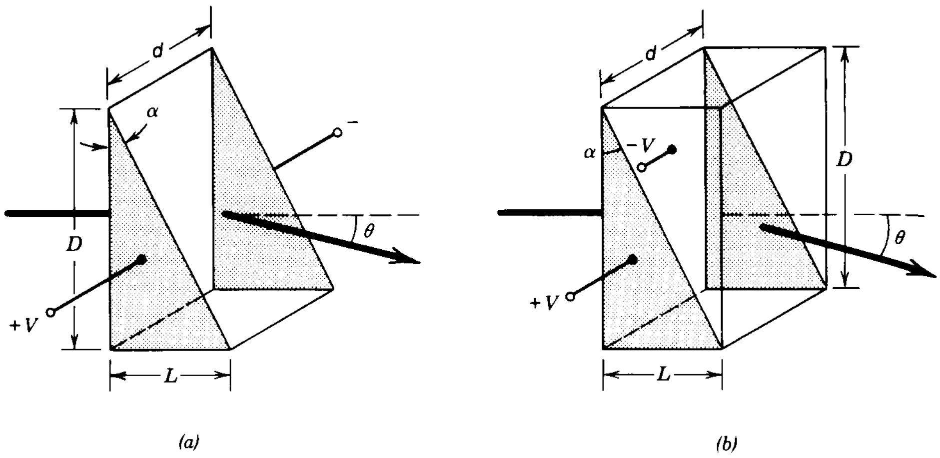


Figure 3: Electro-optic deflection of light.

Then, the phase of the light is modulated like $\Delta\phi = \delta \sin \Omega t$ where the modulation index is given by

$$\delta = \frac{1}{2}\frac{2\pi}{\lambda} n_0^3 r_{63} V_0 = \omega_0 n_0^3 r_{63} V_0/(2c) \tag{9}$$

Omitting the constant phase factor $\omega \ell n_0/c$,the output light field is therefore described by,

$$E_{out} = A\cos(\omega_0 t + \delta \sin \Omega t) \tag{10}$$

which can be written as a series of Bessel functions,

$$\begin{aligned} E_{out} = A[&J_0(\delta)\cos\omega_0 t + J_1(\delta)\left\{\cos(\omega_0+\Omega)t - \cos(\omega_0-\Omega)t\right\} \\ &+ J_2(\delta)\left\{\cos(\omega_0+2\Omega)t + \cos(\omega_0-2\Omega)t\right\} ..] \end{aligned}$$

where the $J_n(\delta)$ represents a Bessel function of the n th order. The result is that the *spectrum* of the output light now contains sidebands shifted by harmonics of the modulator frequency either side of the central carrier (laser) frequency, i.e. $\omega \pm n\Omega$.

Electro-optic deflection of light

An optical beam can be dynamically deflected by electrical control of the refractive index of a prism. The angle of deflection produced by a prism with small apex angle α and refractive index n is $\theta \sim (n-1)\alpha$. A change in refractive index Δn caused by an electric field E corresponds to a change in the deflection angle of

$$\Delta\theta = \alpha\Delta n = -\frac{1}{2}\alpha r n^3 E = -\frac{1}{2}\alpha r n^3 \frac{V}{d} \tag{11}$$

Note here the factor of 1/2 because of the way the crystal is cut: voltage applied along the $z-$direction; light polarised along $x'-$direction (direction of dimension D) and y'-direction coincident with dimension L.

The resolution of the scanner is of course determined by the angular diffraction $\delta\theta \sim \lambda_0/D$ where D is the diameter of the incident laser beam. To minimise this the beam should be as large as possible and fill the aperture of the prism. Unfortunately as d is increased so to must V to maintain the same electric field strength E on the crystal.

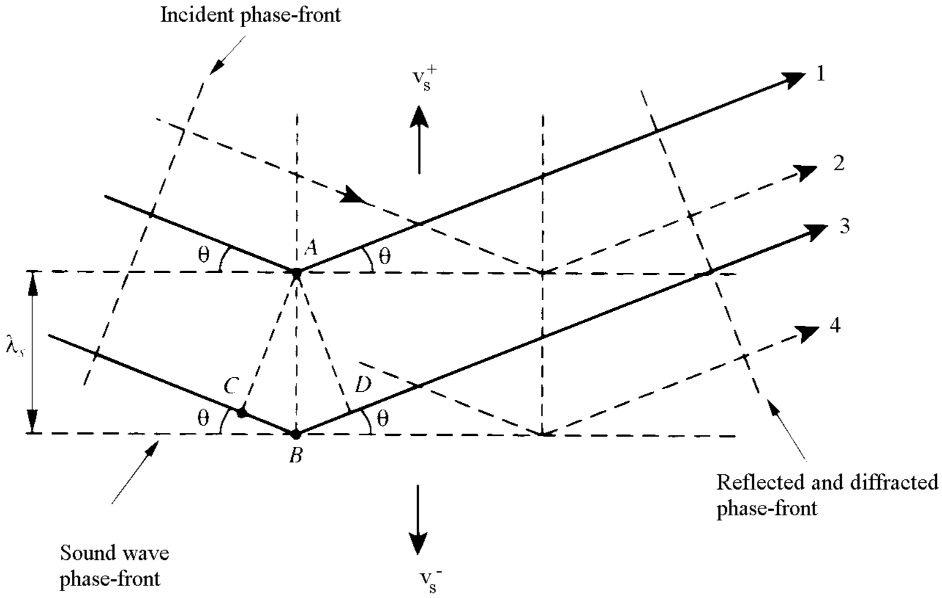


Figure 4: Bragg condition for diffraction from a sound wave.

Acousto-optic deflection of light

We also mention here as an aside that modulation of light beams by sound waves is possible and that acousto-optic devices are commonly used to scan the frequency of a laser beam or to modulate its intensity. Acousto-optic effects come under the headings of: Bragg scattering, Raman-Nath or Debeye-Sear scattering and Brillouin scattering. As an illustration we show below the situation for an acousto-optic device operating in the Bragg regime *i.e.*,where the incident light interacts with broad sound wave, phase-fronts. The relevant Bragg condition is then,

$$\sin\theta = \frac{m\lambda}{2\lambda_s} \tag{12}$$

where λ_s refers to the sound wave and $m = 1$ for a sinusoidal sound wave disturbance. The sound wave moving in the upward direction with velocity v_s^+ Doppler shifts the light wave to frequency $\upsilon + f_s$; similarly the downward propagating sound wave down-shifts the light to frequency $\upsilon - f_s$. Equivalently we may say that an incident photon with wave-vector k_1 collides with a quantised particle of acoustic energy (a phonon) with wave-vector k_s to create a photon of wave-vector k_2 which satisfies momentum conservation through the relationship

$$k_2 = k_1 \pm k_s \tag{13}$$

Harmonic Generation

In lecture 1 (equation 4) we saw that for a certain class of crystals it was possible to have a $\chi^{(2)}$ effect and thus to induce a polarisation oscillating at the second harmonic frequency using two identical laser fields. In the equation below we take account there being two equivalent terms $(E(\omega_1)E(\omega_2) = E(\omega_2)E(\omega_1))$ by inserting the factor of 1/2, thus,

$$P^{2\omega} = \varepsilon_0 \frac{\chi^{SHG}}{2} E^{\omega} E^{\omega} \tag{14}$$

The equation for the second harmonic wave is then

$$\frac{d^2E^{2\omega}}{dz^2} + \left(\frac{2\omega\ n^{2\omega}}{c}\right)^2 E^{2\omega} = -\mu_0(2\omega)^2P^{2\omega} \tag{15}$$

while that for the fundamental wave is

$$\frac{d^2E^{\omega}}{dz^2} + \left(\frac{\omega\ n^{\omega}}{c}\right)^2 E^{\omega} = -\mu_0(\omega)^2P^{\omega} \tag{16}$$

Solutions are of the form

$$E^{\omega} = A^{\omega}\exp[ik^{\omega}z];\ \ E^{2\omega} = A^{2\omega}\exp[ik^{2\omega}z] \tag{17}$$

where $A_{\omega} = A_0(z)\exp(-i\omega t)$

Substituting solutions of this form into 15, and replacing $P^{2\omega}$ using 14 we find the second harmonic amplitude is given by,

$$\frac{dA^{2\omega}}{dz} = \left(\frac{i(2\omega\frac{1}{2}\chi^{SHG})(A^{\omega})^2}{2n^{2\omega}c}\right)\exp(i\Delta kz) \tag{18}$$

where the wave vector mismatch is

$$\Delta k = 2k^{\omega} - k^{2\omega} \tag{19}$$

The intensity (Poynting's vector[1]) is thus

$$I^{2\omega} = \frac{1}{2}n^{2\omega}\sqrt{\frac{\varepsilon_0}{\mu_0}}\left|A^{2\omega}\right|^2 \tag{20}$$

so that integrating 18 we have finally,

$$I^{2\omega} = \frac{(2\omega)^2(\frac{1}{2}\chi^{SHG})^2}{2n^{2\omega}(n^{\omega})^2c^3\varepsilon_0}(I^{\omega})^2\left\{\frac{\sin(\Delta kz/2)}{\Delta kz/2}\right\}^2 z^2 \tag{21}$$

This is a maximum when $z = 2\pi/\Delta k = \ell_c$ which introduces the coherence length ℓ_c

[1] Maxwell's equations lead to the following continuity equation in the absence of any Joule heating term.

$$\boldsymbol{\nabla}.\mathbf{S} + \frac{\partial\rho}{\partial t} = 0$$

Or using the divergence theorem,

$$\int \mathbf{S}.d\mathbf{A} = -\frac{\partial}{\partial t}\int \rho dV$$

where the energy density ρ is given by $\frac{1}{2}\{\mathbf{E}.\mathbf{D}+\mathbf{H}.\mathbf{B}\}$ and Poynting's vector $\mathbf{S}$ is equal to $\frac{1}{2}\,\mathrm{Re}\{\mathbf{E}\times\mathbf{H}^*\}$.Thus the in or outward flow of energy over a surface is equal to the change in the energy density bounded by the surface. For a plane wave in vacuo E and H are related by the intrinsic impedance $Z_0 = \sqrt{\mu_0/\varepsilon_0}$. Note that the optical intensity is equal to the magnitude of the time-averaged Poynting vector, *i.e.*,$I = \langle S\rangle$

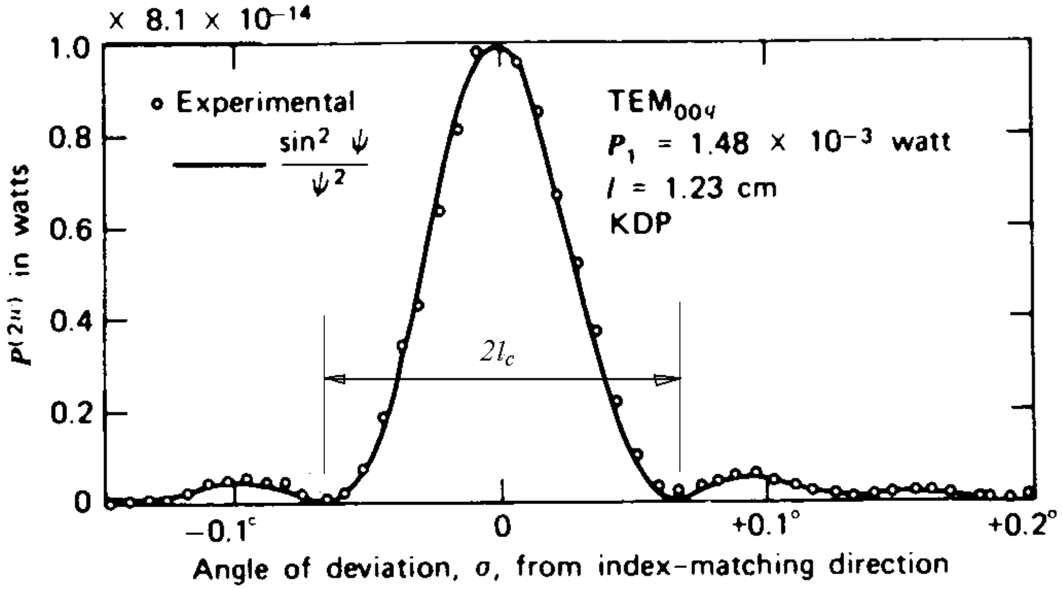


Figure 5: Effect of phase-matching in KDP. The graph shows the variation in SHG ouptut as a function of the phase-match angle.

Appendix A: The missing steps

Start by taking the curl of the curl E equation and include the polarisation in two parts: linear and non-linear

$$\nabla \times (\nabla \times E) = \nabla(\nabla . E) - \nabla^2 E = \nabla \times \left(-\frac{\partial B}{\partial t}\right) \tag{22}$$

$$\nabla^2 E = \frac{\partial}{\partial t}(\nabla \times \mu_0 H) = \frac{\partial}{\partial t}\left(\mu_0 J + \mu_0 \frac{\partial D}{\partial t}\right) \tag{23}$$

$$= \mu_0 \frac{\partial^2}{\partial t^2}(\varepsilon_0 E + P_L + P_{NL}) = \mu_0 \varepsilon_0 \varepsilon_r \frac{\partial^2 E}{\partial t^2} + \mu_0 \frac{\partial^2 P_{NL}}{\partial t^2} \tag{24}$$

$$\nabla^2 E - \mu_0 \varepsilon_0 \varepsilon_r \frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P_{NL}}{\partial t^2} \tag{25}$$

Now make the following substitutions:

$$P_{NL} = \varepsilon_0 \frac{\chi}{2} E_\omega E_\omega; \quad E_\omega = A_\omega e^{ik_\omega z} \tag{26}$$

Take the case of plane waves (1D - case) propagating along z, then the 2ω equation (see equation 15) becomes:

$$\frac{d^2 E_{2\omega}}{dz^2} + \frac{(2\omega)^2 n^2}{c^2} E_{2\omega} = -\mu_0 (2\omega)^2 P_{2\omega} \tag{27}$$

$$-k_{2\omega}^2 A_{2\omega} + 2ik_{2\omega}\frac{dA_{2\omega}}{dz} + \left(\frac{n_{2\omega} 2\omega}{c^2}\right)^2 A_{2\omega} = -\mu_0 (2\omega)^2 \varepsilon_0 \frac{\chi}{2} A_\omega^2 \tag{28}$$

The first and third terms on the LHS of the equation are equal and opposite so that we get:

$$\frac{i\left(2\omega\right)^{2}A_{\omega}^{2}\chi\exp\left(i\Delta kz\right)}{2c^{2}2k_{2\omega}}=\frac{i\left(2\omega\frac{1}{2}\chi^{SHG}\right)A_{\omega}^{2}}{2n_{2\omega}c}\exp\left(i\Delta kz\right)=\frac{dA_{2\omega}}{dz} \tag{29}$$

where we have assumed a small variation of A with z and $\Delta k=2k_{\omega}-k_{2\omega}$. Thus, the equation reduces as follows:

$$\Longrightarrow\int\limits_{0}^{z}\exp\left(i\Delta kz\right)dz\Longrightarrow sinc\ \{\} \tag{30}$$

Finally, noting $I_{2\omega}\propto A_{2\omega}^{2}$ this leads directly to equations 21.