Problem Sets

For each topic, a "Minimal Set" of exercises is offered first, followed by "Supplementary Problems". The latter are not necessarily any more difficult than the former: "supplementary" just means that you can get by without them if you choose to do as little as possible (although I cannot imagine why you would do so, at £9k a year). If you are struggling with time, you might want to do some of these during the vacation. Questions that I consider slightly more challenging are marked with a star. Finally, there are "Extracurricular Problems", which are extracurricular (but not necessarily hard or surplus to requirements for a basic education). Some of these will require independent study of material that may or may not be covered in small font in my lecture notes.

Problem Set 1: First-Order ODEs

Minimal Set

1.1. Determine the order of the following differential equations and whether they are linear or nonlinear. Rewrite them as systems of 1st-order ODEs.

$$\text{(i)}\quad y'' + k^2 y = f(x),$$
$$\text{(ii)}\quad y''' + 2yy' = \sin x,$$
$$\text{(iii)}\quad y' + y^2 = yx.$$

1.2. Solve the following differential equations using the method stated.

(a) Full differential:

$$y' = \frac{(3x^2 + 2xy + y^2)\sin x - 2(3x + y)\cos x}{2(x + y)\cos x}.$$

(b) Separable:

$$\text{(i)}\quad y' = \frac{xe^y}{1 + x^2}, \quad y = 0 \text{ at } x = 0,$$
$$\text{(ii)}\quad y' = \frac{2xy^2 + x}{x^2 y - y}.$$

(c) Reducible to separable by change of variables:

$$y' = 2(2x + y)^2.$$

(d) Homogeneous:

$$2y' = \frac{xy + y^2}{x^2}.$$

(e) Reducible to homogeneous by change of variables:

$$y' = \frac{x + y - 1}{x - y - 2}.$$

(f) Linear:

$$\text{(i)}\quad y' + \frac{y}{x} = 3, \quad x = 0 \text{ at } y = 0,$$
$$\text{(ii)}\quad y' + y\cos x = \sin 2x.$$

(g) Bernoulli:

$$y' + y = xy^{2/3}.$$

1.3. Solve the following 1st-order differential equations:

$$\begin{aligned}
&\text{(i)}\quad y' = \frac{x - y\cos x}{\sin x},\\
&\text{(ii)}\quad (3x + x^2)y' = 5y - 8,\\
&\text{(iii)}\quad y' + \frac{2x}{y} = 3,\\
&\text{(iv)}\quad y' + \frac{y}{x} = 2x^{3/2}y^{1/2},\\
&\text{(v)}\quad 2y' = \frac{y}{x} + \frac{y^3}{x^3},\\
&\text{(vi)}\quad xyy' - y^2 = (x+y)^2 e^{-y/x},\\
&\text{(vii)}\quad x(x-1)y' + y = x(x-1)^2,\\
&\text{(viii)}\quad 2xy' - y = x^2,\\
&\text{(ix)}\quad y' = \cos(y+x), \quad y = \frac{\pi}{2} \text{ at } x = 0,\\
&\text{(x)}\quad y' = \frac{x-y}{x-y+1},\\
&\text{(xi)}\quad y' = \cos 2x - y\cot x, \quad y = \frac{1}{2} \text{ at } x = \frac{\pi}{2},\\
&\text{(xii)}\quad y' + ky = y^n \sin x, \quad n \neq 1.
\end{aligned}$$

Supplementary Problems

1.4. By introducing a new variable $z = 4y - x$, or otherwise, find all solutions of the ODE

$$y' - 16y^2 + 8xy = x^2. \tag{1.1}$$

You should find this solution:

$$y = \frac{x}{4} - \frac{1}{8}\tanh(2x + C). \tag{1.2}$$

Are there any others? Hint: if you divide by zero too recklessly, you may lose some solutions.

1.5. Full Differentials. Solve the following equations, which can be reduced to equations in full differentials.

(a)

$$2\left(x - y^4\right)\mathrm{d}y = y\,\mathrm{d}x, \tag{1.3}$$

Hint: look for an integrating factor (§2.1.1) in the form $\Lambda(y)$.

(b*)

$$y\,(2y\,\mathrm{d}x - x\,\mathrm{d}y) + x^2\,(y\,\mathrm{d}x + 2x\,\mathrm{d}y) = 0. \tag{1.4}$$

Hint: identify within the equation combinations that are full differentials of some functions of x and y and then use those functions to introduce a change of variables that will allow you to separate (the new) variables and integrate.

1.6. Sometimes an ODE can be turned into a homogeneous one (§2.3) by the change of variables $y = z^n$. Solve the following system by this method:

$$y\,\mathrm{d}x + x\,(2xy + 1)\,\mathrm{d}y = 0. \tag{1.5}$$

Hint: you will need to find n such that the equation does become homogeneous; once you have done this, remember that sometimes it is more convenient to look for a solution in the form $x = x(y)$, rather than $y = y(x)$.

1.7. Quasi-homogeneous Equations. (a) An ODE $y' = f(x, y)$ is called *quasi-homogeneous* if $\forall\lambda$ and some $\alpha \neq 0$, $\beta \neq 0$,

$$f(\lambda^\alpha x, \lambda^\beta y) = \lambda^{\beta-\alpha} f(x, y). \tag{1.6}$$

Show that the change of variables $y = x^{\beta/\alpha} z$ reduces a quasi-homogeneous equation to a separable one.

(b) Solve the resulting equation in quadratures. Show that if $\exists z_0$ satisfying $f(1, z_0) = \beta z_0/\alpha$, then $y = z_0 x^{\beta/\alpha}$ is also a solution of the equation. Under what condition are composite solutions (§2.2.1) possible?

You can learn more about quasi-homogeneous equations from Arnold (2006, §§6.4-6.5).

Extracurricular Problems

1.8. Integral Curves and Orthogonal Curves. (a) If a famlily of curves on the plane (x, y) is specified in the form

$$f(x, y, C) = 0, \tag{1.7}$$

where C is a parameter, then one can find a first-order ODE for which these curves are integral curves by taking the differential of (1.7) and then substituting $C = C(x, y)$ extracted from (1.7).

By this method, find the ODE whose integral curves are

$$y = \tan\left[\ln(Cx)\right]. \tag{1.8}$$

(b) Show that the curves that are orthogonal to the integral curves of the ODE

$$F(x, y, y') = 0 \tag{1.9}$$

are integral curves of the ODE

$$F\left(x, y, -\frac{1}{y'}\right) = 0. \tag{1.10}$$

The explicit expression for these *orthogonal curves* can then be found by integrating (1.10).

(c) Find the expression for the curves orthogonal to the family (1.8). Check your answer:

$$3x^2 + 2y^3 + 6y = C. \tag{1.11}$$

1.9. Riccati equations. (a) By the method of §2.6, solve the following Riccati equation

$$y' = y^2 - 2e^x y + e^{2x} + e^x. \tag{1.12}$$

(b) The equation in Q1.4 is also a Riccati equation. Solve it again, but now using the method of §2.6.

1.10. Lagrange's and Clairaut's Equations. (a) *Lagrange's equation* is

$$y = a(y')x + b(y'), \tag{1.13}$$

where $a(p)$ and $b(p)$ are some given continuously differentiable functions. As suggested in §2.7.2, introduce $p = y'$ and rewrite this equation as a first-order ODE with respect to p and x. Find the solution in quadratures, in the form $x(p)$, assuming $a(p) \neq p$.

(b) What if $\exists p_0$ such that $a(p_0) = p_0$? Can you find a special solution of (1.13) that exists in this case?

(c) Now suppose that $a(p) = p \; \forall p$. Then (1.13) is called *Clairaut's*[80] *equation*:

$$y = xy' + b(y'). \tag{1.14}$$

Find all possible solutions of this equation, commenting on the number of continuous derivatives that $b(p)$ must have in order for these solutions to be valid.

(d) Sketch the integral curves of (1.14) if $b(p) = p^2$. They will turn out to be a family of lines hugging a certain envelope. Find the curve that describes this envelope.

(e) Does the solution of (1.14) with $b(p) = p^2$ that passes through the point (x_0, y_0) exist $\forall (x_0, y_0) \in \mathbb{R}^2$? If not, for which (x_0, y_0) does it exist? Given some (x_0, y_0) for which it does exist, is it unique? If not, is the solution unique in the vicinity of (x_0, y_0) if we also specify $y'(x_0) = p_0$? How many legitimate options for p_0 are there at each point? How far can such a solution be continued uniquely? Show graphically using the result of (d) how an infinite number of composite solutions (§2.2.1) can be constructed.

If you are intrigued by Clairaut's equation, you will find a number of interesting facts, examples and applications in Arnold (2006, §8.5) and Tenenbaum & Pollard (1986, §61).

1.11.* Parametric Solutions. Consider an equation formed by y' and x lying on the "folium of Descartes":

$$(y')^3 + x^3 = xy'. \tag{1.15}$$

Try $y' = p = xu$, work out a parametrisation of the curve (x, y'), and then find a parametric solution of (1.15) in the form $x = x(u)$ and $y = y(u)$. In what range of values of u (and, therefore, of x and y) is your solution valid?

[80] Alexis Claude Clairaut FRS (1713-1765), French mathematician, astronomer, Arctic explorer and womaniser.

Problem Set 2: Second-Order ODEs

Minimal Set

2.1. Homogeneous ODEs. Solve these equations:

$$\begin{aligned} &\text{(i)} \quad y'' + 2y' - 15y = 0, \\ &\text{(ii)} \quad y'' - 6y' + 9y = 0, \quad y = 0,\ y' = 1 \text{ at } x = 0, \\ &\text{(iii)} \quad y'' - 4y' + 13y = 0, \\ &\text{(iv)} \quad y''' + 7y'' + 7y' - 15y = 0. \end{aligned}$$

In (iii), write the solution both in terms of complex exponentials *and* in terms of sines and cosines.

2.2. Damped Oscillator. A damped harmonic oscillator is displaced by a distance y_0 and released at time $t = 0$. Show that the subsequent motion is described by the differential equation

$$m\ddot{y} + m\gamma\dot{y} + m\omega_0^2 y = 0 \quad \text{with } y = y_0,\ \dot{y} = 0 \text{ at } t = 0, \tag{2.1}$$

explaining the physical meaning of the parameters m, γ and ω_0.

(a) Find and sketch solutions for (i) overdamping, (ii) critical damping, and (iii) underdamping. (iv) What happens for $\gamma = 0$?

(b) For a lightly damped oscillator ($\gamma \ll \omega_0$), the quality factor, or Q-factor, is defined as

$$Q = 2\pi \frac{\text{energy stored}}{\text{energy lost per period of oscillation}}. \tag{2.2}$$

Show that $Q = \omega_0/\gamma$.

2.3. Boundary-Value Problem. (a) Solve the ODE

$$y'' + k^2 y = 0. \tag{2.3}$$

(b) Consider the above equation on the interval $x \in [0, L]$. Are there solutions that are not identically zero everywhere but have $y(0) = y(L) = 0$? For which values of k do such solutions exist? Find these solutions and explain in terms of linear algebra the meaning of what you have done.

(c*) Can any function that is defined in the interval $[0, L]$ and vanishes at its ends be represented as a linear combination of solutions of (2.3) with different k's? Work out how to do it.

The full theory of boundary-value problems is not covered in this course. Read up on the topic or wait till the Mathematical Methods course in your 2nd year (see Eßler 2009, §25 and Lukas 2018, §5)—although you will, in fact, need to solve (2.3) many times in CP4.

2.4. Inhomogeneous ODEs. Consider the equation

$$y'' - 3y' + 2y = f(x). \tag{2.4}$$

What is its particular solution for

$$f(x) = \begin{array}{rl} \text{(i)} & x^2, \\ \text{(ii)} & e^{4x}, \\ \text{(iii)} & e^x, \\ \text{(iv)} & \sinh x, \\ \text{(v)} & \sin x, \\ \text{(vi)} & x \sin x, \\ \text{(vii)} & e^{2x} + \cos^2 x. \end{array}$$

2.5. Inhomogeneous ODEs. Solve these equations:

$$\begin{array}{rl} \text{(i)} & 5y'' + 2y' + y = 2x + 3, \quad y = -1,\ y' = 0 \text{ at } x = 0, \\ \text{(ii)} & y'' - y' - 2y = e^{2x}, \\ \text{(iii)} & 4y'' - 4y' + y = 8e^{x/2}, \quad y = 0,\ y' = 1 \text{ at } x = 0, \\ \text{(iv)} & y'' + 3y' + 2y = xe^{-x}, \\ \text{(v)} & y'' - 4y' + 3y = 10 \cos x, \\ \text{(vi)} & y'' + 4y = x + \cos 2x, \quad y = 0 \text{ when } x = 0, \\ \text{(vii)} & y'' - 2y' + 2y = e^x(1 + \sin x), \quad y = 0 \text{ at } x = 0 \text{ and at } x = \dfrac{\pi}{2}, \\ \text{(viii)} & y'' + 2y' + y = 2e^{-x} + x^3 + 2 \cos x, \\ \text{(ix)} & y'' - 2y' + y = 3e^x, \quad y = 3,\ y' = 0 \text{ at } x = 0, \\ \text{(x)} & x^2 y'' + xy' + y = x. \end{array}$$

2.6. Forced Oscillator. When a varying couple $I \cos \omega t$ is applied to a torsional pendulum with natural period $2\pi/\omega_0$ and moment of inertia I, the angle of the pendulum obeys the equation of motion

$$\ddot{\theta} + \omega_0^2 \theta = \cos \omega t. \tag{2.5}$$

The couple is first applied at time $t = 0$ when the pendulum is at rest in equilibrium.

(a) Show that, in the subsequent motion, the root-mean-square angular displacement is $1/|\omega_0^2 - \omega^2|$ when the average is taken over a time large compared to $1/|\omega_0 - \omega|$.

(b) Discuss the motion as $|\omega_0 - \omega| \to 0$.

2.7. Forced and Damped Oscillator. Consider the damped oscillator of Q2.2 subject to an oscillatory driving force:

$$m\ddot{y} + m\gamma\dot{y} + m\omega_0^2 y = F \cos \omega t. \tag{2.6}$$

(i) Explain what is meant by the stationary solution of this equation, and calculate this solution for the displacement $y(t)$ and the velocity $\dot{y}(t)$.

(ii) Sketch the amplitude and phase of $y(t)$ and $\dot{y}(t)$ as a function of ω.

(iii) Determine the resonant frequency for both the displacement and the velocity.

(iv) Defining $\Delta\omega$ as the full width at half maximum of the resonance peak calculate $\Delta\omega/\omega_0$ to leading order in γ/ω_0.

(v) For a lightly damped, driven oscillator near resonance, calculate the energy stored and the power supplied to the system. Hence confirm that $Q = \omega_0/\gamma$ as in Q2.2. How is Q related to the width of the resonance peak?

2.8. Verify that $y = x + 1$ is a solution of

$$(x^2 - 1)y'' + (x + 1)y' - y = 0. \tag{2.7}$$

Hence find the general solution of this equation. Check your answer:

$$y = C_1(x + 1) + C_2\left(\frac{x + 1}{4}\ln\left|\frac{x - 1}{x + 1}\right| + \frac{1}{2}\right). \tag{2.8}$$

2.9. Consider the differential equation

$$x(x + 1)y'' + (2 - x^2)y' - (2 + x)y = (x + 1)^2. \tag{2.9}$$

(a) One of its homogeneous solutions is $y_1(x) = 1/x$. Find the general solution.

(b) Now pretend that you do not know that $1/x$ is a homogeneous solution, but know the second homogeneous solution, $y_2(x)$, that you found in (a) (in fact, if you stare at the equation for a few seconds, or minutes, you will see that you could have guessed that solution). Use the knowledge of $y_2(x)$ to find both $y_1(x)$ and the general solution of the equation.

2.10. Nonlinear ODEs. All of this problem set so far has been a drill in solving linear, second-order ODEs. There are few general methods for solving nonlinear ones (and most of them cannot be integrated in quadratures anyway). Still, there are a few tricks, which you now have an opportunity to practice. All of them, naturally, are based on methods for lowering the order of the equation down to first.

(a) If bits of the equation can be manipulated into full derivatives of some expressions, the equation's order can sometimes be lowered by direct integration. Practice this by solving the ODE

$$1 + yy'' + (y')^2 = 0. \tag{2.10}$$

(b) If the equation does not contain y, only its derivatives, the order is lowered by treating the lowest-order derivative that does appear as the new function. Practice this by solving the ODE

$$xy'' = y' + (y')^3. \tag{2.11}$$

(c) If the equation does not contain x, you can lower its order by letting $p = y'$ and looking for solutions in the form $p = p(y)$. Note that then, by chain rule, $y'' = pp'$. Solve by this method (plus other appropriate changes of variable) the ODE

$$y''(y - 1) + y'(y - 1)^2 = (y')^2. \tag{2.12}$$

In Q2.15, you will learn two other methods for solving nonlinear ODEs.

Supplementary Problems

2.11. Solve the differential equation

$$y'' - 2y' + (\lambda^2 + 1)y = e^x \sin^2 x \tag{2.13}$$

for general values of the real parameter λ. Identify any special values of λ for which your solution fails and solve the equation also for those values.

2.12. Find the continuous solution with continuous first derivative of the equation

$$y'' + 2y' + 2y = \sin x + f(x), \quad \text{where} \quad f(x) = \begin{cases} 0, & x \leqslant 0, \\ x^2, & x > 0, \end{cases} \tag{2.14}$$

subject to $y(-\pi/2) = y(\pi) = 0$. Hint: obtain a general solution for each of the cases $x < 0$ and $x > 0$ and then determine any loose constants by making these solutions agree at $x = 0$.

2.13. Oscillator with Modulated Force. A mass m is constrained to move in a straight line and is attached to a spring of strength $m\omega_0^2$ and a dashpot which produces a retarding force $m\gamma v$, where v is the velocity of the mass and $\gamma \ll \omega_0$. An amplitude-modulated periodic force $mA\cos\sigma t \sin\omega t$ with $\sigma \ll \omega$ and $\omega = \omega_0$ is applied to the mass. Show that, in the long-time limit, the displacement is an amplitude-modulated wave

$$y = -\frac{A\sin(\sigma t + \phi)\cos\omega t}{2\omega\sqrt{\sigma^2 + \gamma^2/4}}, \quad \tan\phi = \frac{\gamma}{2\sigma}. \tag{2.15}$$

2.14.* Consider the differential equation

$$9xy'' + (6 + x)y' + \lambda y = 0. \tag{2.16}$$

There are several values of λ for which this can be solved via reduction to a 1st-order equation. For at least one of them, it is possible to find a solution that

(i) satisfies $y(x \to \pm\infty) \to 0$, but is not zero everywhere,

(ii) is continuous at $x = 0$.

Experiment with solutions corresponding to various tractable values of λ and find one for which the above two properties are satisfied. The solution you find may, but does not have to, be

$$y = C\, e^{-x/9} \int_{-\infty}^{x} dz\, \frac{e^{z/9}}{|z|^{2/3}}. \tag{2.17}$$

2.15. Homogeneous Nonlinear ODEs. Continuing from Q2.10, here are some more methods for lowering the order of nonlinear ODEs.

(a) An equation is called *homogeneous* with respect to y if it does not change under the rescaling $y \to \lambda y$ $\forall\lambda \neq 0$ (i.e., every term contains the same number of powers of y or its derivatives). For such an equation, one can lower the order by introducing a new function $z(x)$ via $y' = yz$. Then $y'' = y(z' + z^2)$, so the second derivative is expressible in terms of the first; powers of y will cancel because the equation is homogeneous.

Solve by this method the following ODE:

$$xyy'' + x(y')^2 = 3yy'. \tag{2.18}$$

Check your answer: $y^2 = Ax^4 + B$.

(b*) This equation can also be solved by the method that you practiced in Q2.10(a), if you can spot the full derivatives. Do it.

(c) A generalised version of (a) is an equation that does not change under the rescaling $x \to \lambda x$, $y \to \lambda^\alpha y$ $\forall\lambda \neq 0$ and for some α (the linear version of this with $\alpha = 0$ is Euler's equation, §5.1.4; you encountered the first-order case in Q1.7). This is solved by letting $x = e^t$ when $x > 0$, $x = -e^t$ when $x < 0$, and $y = z(t)e^{\alpha t}$. The result will be an equation for z in which t does not appear explicitly, so you can use the method practiced in Q2.10(c).

Consider the following ODE:

$$x^2(y'' + 2yy') + 2xy^2 - 2y = 0, \quad x > 0 \tag{2.19}$$

and establish for what value of α it has the invariance property described above. Then solve the equation by the proposed method.

EXTRACURRICULAR PROBLEMS

2.16. Higher-Order Linear ODEs with Constant Coefficients. Solve the following ODEs

$$\text{(i)}\ \ y'''' - 6y''' + 8y'' + 6y' - 9y = 0,$$
$$\text{(ii)}\ \ y'''' + 4y'' + 4y = 0.$$

Hint: in (i), you will need to guess some roots of a 4th-order polynomial and then factorise it.

2.17.* Series Solution of Linear ODEs. ODEs (usually second order) that cannot be solved in quadratures can sometimes be solved in terms of series. One can then give these solutions names and establish all their properties, thus expanding the library of functions that we know how to handle. Such functions are called *special functions* (a classic textbook on them is Lebedev 1972).

Consider *Airy's*[81] *equation*

$$y'' + xy = 0. \tag{2.20}$$

This equation cannot, alas, be solved in quadratures. Seek its solution in the form of a power series

$$y(x) = \sum_{n=0}^{\infty} c_n x^n. \tag{2.21}$$

(a) Without worrying about convergence of this series or about the legitimacy of differentiating it term by term, work out what equations the coefficients c_n must satisfy in order for (2.21) to be a solution of (2.20).

(b) Find two linearly independent such series solutions (remember that if they are linearly independent at one point, they are linearly independent everywhere; see §4.3). Use d'Alembert's[82] Ratio Test to show that both series converge. What is the general solution of (2.20)?

Certain linear combinations of the solutions that you have found are called *Airy functions*, an example of special functions. Entire books are written about them (Vallée & Soares 2004).

(c) Find the general solution of (2.3) of Q2.3 (with $k = 1$ for simplicity) by the same method and convince yourself that the series that you have obtained are sines and cosines. These functions too were once special.

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[81] Sir George Biddell Airy KCB FRS (1801-1892) was Astronomer Royal and put the Prime Meridian at Greenwich.

[82] Jean-Baptiste le Rond d'Alembert (1717-1783), great French mathematician, physicist, philosopher, musical theorist, Diderot's fellow encyclopedist and full-time admirer of Mme de Lespinasse.

It is not always possible to find solutions in the form of a Taylor series (2.21). For example, the equation

$$x^2 y'' + xp(x)y' + q(x)y = 0, \tag{2.22}$$

where $p(x)$ and $q(x)$ are representable as convergent Taylor series, is only guaranteed to have one solution in the form of a *Frobenius*[83] *series*

$$y(x) = x^\alpha \sum_{n=0}^{\infty} c_n x^n, \tag{2.23}$$

where α is not, in general, a (positive) integer; you can always get the second solution via the "buy one get one free" scheme (§4.6). You will encounter the Frobenius method in the Mathematical Methods course (see Eßler 2009, §23 or Lukas 2018, §5.3 for some examples; if you want to learn the general method, see, e.g., White 2010, §4 or Bender & Orszag 1999, §3; see also Coddington 1990, Ch. 4). However, no one stops you from getting intrigued now and at least convincing yourself that (2.23) works for (2.22) and that (2.21) in general does not. A good example of (2.22) to play with is *Bessel's*[84] *equation:*

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0, \tag{2.24}$$

where ν is a real number (interesting things happen depending on whether it is an integer). On Bessel functions too, there are entire books in the library (short and elementary: Bowman 2003; long and comprehensive: Watson 1944).

[83] Ferdinand Georg Frobenius (1849-1917), German mathematician, student of Weierstrass.

[84] Friedrich Wilhelm Bessel (1784-1846), a fairly boring German mathematician and astronomer. Bessel's functions were in fact discovered by Daniel Bernoulli, nephew of Jacob Bernoulli of Bernoulli's equation.

Problem Set 3: Systems of ODEs

(vacation work)

Minimal Set

3.1. Solve the coupled differential equations

$$\begin{cases} \dot{x} + ax - by = f, \\ \dot{y} + ay - bx = 0, \end{cases} \tag{3.1}$$

where a, b, and f are constants.

3.2. Solve the coupled differential equations

$$\begin{cases} y' + 2z' + 4y + 10z - 2 = 0, \\ y' + z' + y - z + 3 = 0, \end{cases} \tag{3.2}$$

where $y = 0$ and $z = -2$ at $x = 0$.

3.3. Find the general, *real* solutions of the following homogeneous systems of ODEs

$$\text{(i)} \begin{cases} \dot{x} = -2y + 2z, \\ \dot{y} = x - y + z, \\ \dot{z} = y - z, \end{cases}$$

$$\text{(ii)} \begin{cases} \dot{x} = 4x - y - z, \\ \dot{y} = x + 2y - z, \\ \dot{z} = x - y + 2z. \end{cases}$$

3.4. Find the general, *real* solutions of the following inhomogeneous systems of ODEs

$$\text{(i)} \begin{cases} \dot{x} = 4x + 3y - 3z, \\ \dot{y} = -3x - 2y + 3z, \\ \dot{z} = 3x + 3y - 2z + 2e^{-t}, \end{cases}$$

$$\text{(ii)} \begin{cases} \dot{x} = -5x + y - 2z + \cosh t, \\ \dot{y} = -x - y + 2\sinh t + \cosh t, \\ \dot{z} = 6x - 2y + 2z - 2\cosh t. \end{cases}$$

Supplementary Problems

3.5. Solve the coupled differential equations

$$\begin{cases} 2y'' - 3y' + 2z' + 3y + z = e^{2x}, \\ y'' - 3y' + z' + 2y - z = 0. \end{cases} \tag{3.3}$$

Is it possible to have a solution to these equations for which $y = z = 0$ at $x = 0$? What is wrong with this system?

3.6. Charged Particle in Electromagnetic Field. A particle of mass m and charge q is placed, initially at rest, in straight, constant electric $\boldsymbol{E}$ and magnetic $\boldsymbol{B}$ fields, which are neither perpendicular nor parallel to each other. It will experience the Lorentz force

$$\boldsymbol{F} = q\left(\boldsymbol{E} + \frac{\boldsymbol{v} \times \boldsymbol{B}}{c}\right), \tag{3.4}$$

where $\boldsymbol{v}$ is the particle's velocity. Find $\boldsymbol{v}(t)$ and sketch the particle's motion.

3.7. Non-diagonalisable systems. Find the general solutions of the following homogeneous systems of ODEs

$$\text{(i)} \quad \begin{cases} \dot{x} = 2x + y, \\ \dot{y} = 4y - x, \end{cases}$$

$$\text{(ii)} \quad \begin{cases} \dot{x} = x - 2y, \\ \dot{y} = -x - y - 2z, \\ \dot{z} = y + z, \end{cases}$$

$$\text{(iii)} \quad \begin{cases} \dot{x} = 2x - 5y - 8z, \\ \dot{y} = 7x - 11y - 17z, \\ \dot{z} = -3x + 4y + 6z. \end{cases}$$

Extracurricular Problems

3.8. Duffing's[85] Oscillator. Consider the following nonlinear oscillator:

$$\ddot{x} + \gamma\dot{x} + x + \alpha x^2 + \beta x^3 = 0. \tag{3.5}$$

With $\alpha = 0$, this describes a (damped) oscillator whose restoring force strengthens ($\beta > 0$) or weakens ($\beta < 0$) with amplitude—these are called the cases of *hardening spring* or *softening spring*. When $\alpha \neq 0$, there is another amplitude-dependent force, which pushes/pulls at the oscillator differently depending on the direction of the displacement.

Sketch the phase portraits of this system for the following cases:

$$\begin{aligned} &\text{(i)} \quad \gamma = 0, \quad \alpha = 0, \quad \beta > 0, \\ &\text{(ii)} \quad \gamma = 0, \quad \alpha = 0, \quad \beta < 0, \\ &\text{(iii)} \quad \gamma = 0, \quad 0 < \alpha \ll \sqrt{|\beta|}, \quad \beta < 0. \end{aligned}$$

Having done that, deduce what happens in each of these cases when $0 < \gamma \ll 1$. The name of the game is to get a qualitatively adequate sketch with as little work as possible.

If you liked this so much that you want to play a bit more, look at the case when the sign of the x term is reversed (i.e., linearly, instead of a restoring force, there is an instability).

You can read more about this kind of nonlinear oscillators in Strogatz (1994, §§7.6, 12.5), Landau & Lifshitz (1976, §29) and Glendinning (1994, §7.2).

3.9. Wiggly[86] Pendulum. Consider a nonlinear pendulum whose point of supension rapidly oscillates in the horizontal direction as $a \cos \omega t$ (see Fig. 17b of §5.4), where $\omega \gg \sqrt{g/l}$ and $a \ll l$ (l is the length of the pendulum, g is the acceleration of gravity). Determine its equilibria and the conditions under which they are stable. Sketch the phase portrait for the pendulum's motion averaged over the rapid oscillations.

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[85] Georg Duffing (1861-1944), a German engineer, famous for his oscillator, which, with the sign of the x term reversed and with a periodic external force added, is one of the simplest known systems that can exhibit chaotic behaviour.

[86] Sir William Wiggly FRS (1918-2018), a prominent British vacillator.

3.10. How Nonlinearity Takes Over. Consider the nonlinear systems

$$\text{(i)} \quad \begin{cases} \dot{x} = -y + \alpha x(x^2 + y^2), \\ \dot{y} = x + \alpha y(x^2 + y^2), \end{cases}$$

$$\text{(ii)} \quad \begin{cases} \dot{x} = -x - \dfrac{\alpha y}{\ln(x^2 + y^2)}, \\ \dot{y} = -y + \dfrac{\alpha x}{\ln(x^2 + y^2)}, \end{cases}$$

where α is a (real) parameter. Determine the nature of the fixed point at $(x, y) = 0$. Does the answer survive the restauration of the nonlinearity? Solve the nonlinear equations exactly and determine the nature of the fixed point depending on the value of α.

Hint. Going to the complex plane in the vein of (7.25) (§7.1.6) might help shorten calculations.

3.11. Limit Cycles. Consider the nonlinear systems

$$\text{(i)} \quad \begin{cases} \dot{x} = x - (x - y)\sqrt{x^2 + y^2}, \\ \dot{y} = y - (x + y)\sqrt{x^2 + y^2}, \end{cases}$$

$$\text{(ii)} \quad \begin{cases} \dot{x} = y + \dfrac{1}{4}\, x(1 - 2x^2 - 2y^2), \\ \dot{y} = -x + \dfrac{1}{2}\, y(1 - x^2 - y^2). \end{cases}$$

Sketch their phase portraits. Show, in particular, that there is a stable limit cycle in both cases. In (i), you should be able to derive the cycle explicitly. In (ii), you will need to construct a trapping region.

3.12. Odell's Predator-Prey Model. Consider the following system of nonlinear, coupled ODEs:

$$\begin{cases} \dot{x} = x[x(1 - x) - y], \\ \dot{y} = y(x - a), \end{cases} \tag{3.6}$$

where $a > 0$ is a parameter (Odell 1980). This model describes a vegetarian species x, who are omnigamous (birth rate $\propto x^2$), limited by the availability of food (death rate $\propto x^3$), and are eaten by a predator species y at the rate proportional to the latter's population size; the predators are monogamous but procreate conditionally on the availability of prey, at a rate proportional to x, and have a death rate controlled by the parameter a. The population sizes x and y are normalised in some appropriate way (x to its value corresponding to the equilibrium in the absence of predator).

Determine the equilibria of the system and their nature depending on the value of the parameter a. Sketch the phase portrait for each qualitatively different parameter regime. What is the condition for a limit cycle to exist? Interpret your qualitative solutions in terms of population dynamics.