

FIRST YEAR MATHS FOR PHYSICS STUDENTS

NORMAL MODES AND WAVES

Hilary Term 2014

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Question Sheet 2: Waves 1

1. At time $t = 0$, the displacement of an infinitely long string is defined as:

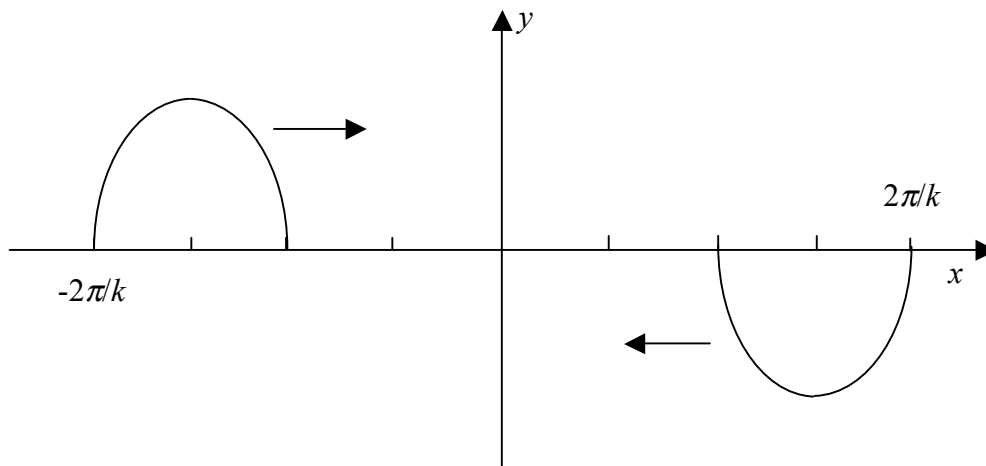
$$y(x, t = 0) = \sin \frac{\pi x}{a} \text{ in the range } -a \leq x \leq a$$

and $y(x, t = 0) = 0$ otherwise.

The string is initially at rest.

Using d'Alembert's solution, and assuming that waves may move along the string with speed c , sketch the displacement of the string at $t = 0$, $t = a/2c$, and $t = a/c$.

- 2.



Two transverse waves are on the same piece of string. The first has displacement y non-zero only for $kx + \omega t$ between π and 2π , when it is equal to $A \sin(kx + \omega t)$. The second has $y = A \sin(kx - \omega t)$ for $kx - \omega t$ between -2π and $-\pi$, and is zero otherwise. When $t = 0$, the displacement is as shown in the figure. Calculate the energy of the two waves.

What is the displacement of the string at $t = 3\pi/2\omega$? Calculate the energy at this time.

- 3.(a) What is the difference between a moving wave and a stationary wave?
 (b) Convince yourself that

$$y_1 = A \sin(kx - \omega t)$$

corresponds to a moving wave. Which way does it move? What are the amplitude, wavelength, frequency, period and velocity of the wave?

- (c) Show that y_1 satisfies the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2}$$

provided that ω and k are suitably related.

- (d) Write down a wave y_2 of equal amplitude travelling in the opposite direction. Show that $y_1 + y_2$ can be written in the form

$$y_1 + y_2 = f(x)g(t)$$

where $f(x)$ is a function of x only, and $g(t)$ is a function just of t . Convince yourself that the combination of two moving waves is a stationary wave. By determining $f(x)$ and $g(t)$ explicitly, determine the wavelength and frequency of $y_1 + y_2$. Comment on the velocity of the waves.

4. What is meant by (a) a dispersive medium, and (b) phase velocity, v ? Explain the relevance of group velocity g for the transmission of signals in a dispersive medium. Justify the equation

$$g = \frac{d\omega}{dk} \quad (1)$$

Show that alternative expressions for g are

$$g = v + k \frac{dv}{dk} \quad (2)$$

$$g = v - \lambda \frac{dv}{d\lambda} \quad (3)$$

$$g = \frac{c}{\mu} \left(1 + \frac{\lambda}{\mu} \frac{d\mu}{d\lambda} \right) \quad (4)$$

where μ is the refractive index for waves of wavelength λ and wavenumber k (in the medium).

Use Eq. 3 to show that

$$g = v \left[1 - 1 / \left(1 + \frac{v}{\lambda'} \frac{d\lambda'}{dv} \right) \right]$$

where λ' is the wavelength in vacuum.

5. In quantum mechanics, a particle of momentum p and energy E has associated with it a wave of wavelength λ and frequency f given by

$$\lambda = h / p \text{ and } f = E / h$$

where h is Planck's constant. Find the phase and group velocities of these waves given that

$$p = m_0 v / \sqrt{1 - v^2 / c^2} \text{ and } E = m_0 c^2 / \sqrt{1 - v^2 / c^2},$$

(The particle's rest mass is m_0 , and its speed is v . c is the speed of light)

Comment on your answers.

6. Show that the kinetic energy U and the potential energy V for a length $\lambda = 2\pi/k$ of a transverse wave on a string of linear density ρ and at tension T are given by

$$U = \int_0^\lambda \frac{1}{2} \rho \left(\frac{\partial y}{\partial t} \right)^2 dx$$

$$\text{and } V = \int_0^\lambda \frac{1}{2} T \left(\frac{\partial y}{\partial x} \right)^2 dx$$

Evaluate these for the wave

$$y = A \cos(kx + \omega t + \phi)$$

and show that $U = V$.