University of Oxford

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Hilary term
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CP4: Mathematical Methods 2

Additional homework problem

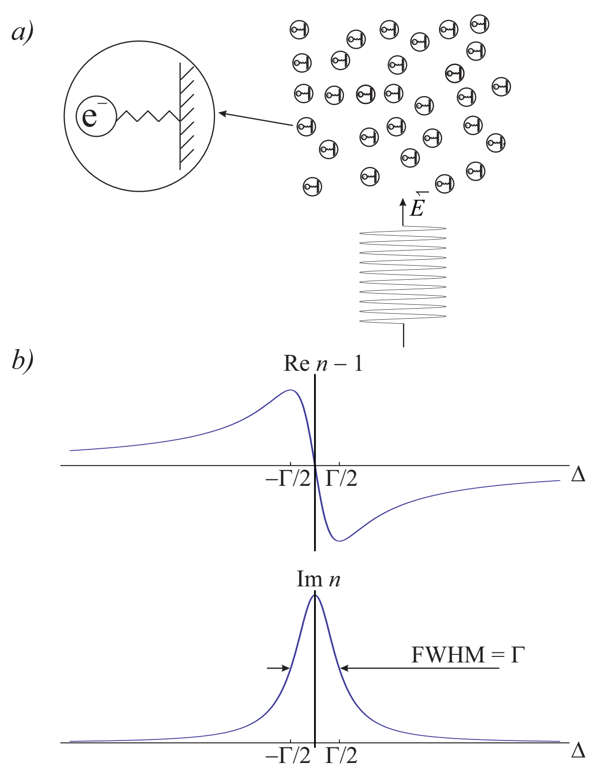

Figure 1: Classical theory of dispersion. (a) Model; (b) Real and imaginary parts of the refractive index.

Problem B.1. A light wave propagating along the z axis inside a linear medium (e.g., a gas of atoms) is governed by the equation

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) E = \frac{1}{\epsilon_0 c^2}\frac{\partial^2}{\partial t^2} P, \tag{1}$$

where P is the *polarization* of the medium, i.e. the density, per unit volume, of the dipole moment induced in the medium by the field. The solution to this equation is

$$E(z,t) = \mathrm{Re} E_0 e^{ikz - i\omega t}. \tag{2}$$

The polarization can be calculated as shown in Fig. 1(a). We treat each atom as a damped harmonic oscillator driven by the oscillating field, where ω_0 is the atomic transition frequency and γ its width.

a) Find the displacement of the electron $x(z,t)$ for an "atom" located at z. **Hint:** don't forget that the electron's charge is negative.

b) Write down the polarization $P(z,t)$ given that the dipole moment of each atom is $-ex(z,t)$ and the number density (number per unit volume) of atoms is N. Simplify your answer under the assumption $\gamma, \Delta \ll \omega_0$ for $\Delta = \omega - \omega_0$. **Hint:** under this assumption, you can also replace $i\omega\gamma \to i\omega_0\gamma$ in the damping term.

c) Find the wavenumber $k(\omega)$ as a function of the frequency ω such that Eq. (2) is a solution to Eq. (1). **Hint:** $k(\omega)$ is significantly complex.

d) Find the refractive index $n(\omega) = \frac{c}{v_{\rm ph}} = \frac{ck}{\omega}$. Check that your result is consistent with Fig. 1(b).

The remainder of the problem should be solved under the assumptions that $|n-1| \ll 1$ for all frequencies (i.e. the absolute effect of the atomic gas on the continuous electromagnetic wave is very small).

e) Show that the field intensity (power per unit area) behaves as $I(z) \propto e^{-\alpha z}$ (Beer's law). The quantity α in this equation is called the absorption coefficient. Find it for $\omega = \omega_0$. **Hint:** the intensity is proportional to the square of the amplitude's absolute value.

f) Find the group velocity at $\omega = \omega_0$. Find the conditions under which the group velocity is (i) superluminal and (ii) negative. **Hint:** it is more convenient to calculate $\frac{1}{v_{\rm gr}} = \frac{\mathrm{d}\mathrm{Re}k}{\mathrm{d}\omega}$.

g) Suppose you wish to perform an experiment to observe negative group velocity in atomic rubidium vapor. Atomic rubidium has a transition with the wavelength of $\lambda = 795$ nm and the linewidth of $\gamma/2\pi = 5.7$ MHz. Find the number density required. Check that the condition $|n-1| \ll 1$ is satisfied at this number density.

h) For the experiment, it is optimal to use a pulse with the duration $\tau \sim \gamma^{-1}$ (so the group velocity is approximately constant over the pulse's spectrum). Justify this choice. For $v_{\rm gr}^{-1} = 0$, estimate the length of the vapor cell such that the absolute value of the group delay, in comparison with a freely propagating pulse, is equal to the pulse duration. Estimate the fractional loss of the pulse's energy after traversing the cell.

See L. J. Wang, A. Kuzmich and A. Dogariu, Gain-assisted superluminal light propagation, Nature **406**, 277–279 (2000) (`https://www.nature.com/articles/35018520`) for an experimental study on this subject.