Quantum Club

Assignment 4

ADJOINT OPERATORS.

- Study Sec. A.7 of the textbook and solve the exercises therein.

Problem 1. Let $\hat{A} \simeq \begin{pmatrix} 1 & 2i \\ 3i & 4 \end{pmatrix}$ and $|a\rangle \simeq \begin{pmatrix} 5 \\ 6i \end{pmatrix}$.

a) Find the matrix of $|b\rangle = \hat{A}|a\rangle$.

b) Find the matrices of $\hat{A}^\dagger$ and $\langle a|$. Multiply them together to find the matrix of $\langle a|\hat{A}^\dagger$ and check if it equals that of $\langle b|$.

FINDING EIGENVALUES AND EIGENVECTORS.

- Study Sec. A.8 of the textbook and solve the exercises A.61, A.62, A.64, A.68 and A.70.

Problem 2. For a 2×2 matrix $\hat{A}$ there exists a nonzero vector $|v\rangle$ such that $\hat{A}|v\rangle = 0$. Show that

$$\det \hat{A} \equiv \left|\hat{A}\right| \equiv A_{11}A_{22} - A_{12}A_{21} \tag{1}$$

vanishes for this matrix.

The expression (1) is called the *determinant* of the matrix $\hat{A}$. It is an important object with many interesting properties, which you will study in your linear algebra class at college. At the moment, we will use the property derived in Problem 2 to find eigenvalues and eigenvectors of matrices — a skill that is critically important in quantum physics.

For an eigenvector $|v\rangle$ with the eigenvalue λ of a matrix $\hat{A}$, we have $\hat{A}|v\rangle = \lambda|v\rangle$, which we can rewrite as $\left(\hat{A} - \lambda\hat{\mathbf{1}}\right)|v\rangle = 0$, which means, in turn, that the determinant $\left|\hat{A} - \lambda\hat{\mathbf{1}}\right| = 0$. This is called the *characteristic equation* of matrix $\hat{A}$. This equation is quadrtatic for 2×2 matrices, and we can solve it to find the two eigenvalues $\lambda_{1,2}$. Then we can substitute them into $\hat{A}|v\rangle = \lambda|v\rangle$ to find $|v\rangle$. If $\hat{A}$ is Hermitian and you do everything right, you will find the vectors $|v_{1,2}\rangle$ to be orthogonal to each other. Don't forget to normalize them if needed!

This video `https://www.youtube.com/watch?v=PFDu9oVAE-g` gives a nice overview of the topic.

Problem 3. Solve the end-of-chapter problem 1.10(a,b). Then repeat the calculation for the matrices

- $\hat{A} \simeq \begin{pmatrix} 9 & 2 \\ 2 & 6 \end{pmatrix}$;
- the three matrices of Ex. A.65.

  Hint: For a three-dimensional matrix, the determinant is
  $\det \hat{A} = A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{11}A_{23}A_{32} - A_{12}A_{21}A_{33} - A_{13}A_{22}A_{31}$.

OBSERVABLES.

- Study Secs. 1.9.1 and 1.9.2 and solve the exercises therein.

Problem 4. End-of-chapter problem 1.10(c-e).

Problem 5. End-of-chapter problem 1.11.

Problem 6. End-of-chapter problem 1.9(a). Solve in the canonical basis. Also write the matrix of the corresponding observable with the eigenvalue of 1 associated with the photon having been transmitted through the PBS and -1 with it having been reflected.

COMMUTATORS AND UNCERTAINTY PRINCIPLE.

- Study Secs. A.9 and 1.9.3 and solve the exercises therein.

Problem 7. Find the commutators

a) $[\hat{\sigma}_z + i\hat{\sigma}_y, \hat{\sigma}_z - i\hat{\sigma}_y]$;

b) $[\hat{\sigma}_x^2 + \hat{\sigma}_y^2 + \hat{\sigma}_z^2, \hat{\sigma}_x]$.

Hint: $[\hat{\sigma}_x, \hat{\sigma}_y] = 2i\hat{\sigma}_z$; $[\hat{\sigma}_y, \hat{\sigma}_z] = 2i\hat{\sigma}_x$; $[\hat{\sigma}_z, \hat{\sigma}_x] = 2i\hat{\sigma}_y$.

Problem 8. Write the uncertainty principle for the observables $A = |H\rangle\langle H| + 3\,|H\rangle\langle V| + 3\,|V\rangle\langle H| + 2\,|V\rangle\langle V|$ and $\hat{B} = |R\rangle\langle R| + 3\,|L\rangle\langle L|$ and state $|\psi\rangle = |+\rangle$. Verify explicitly that it holds.

Problem 9. End-of-chapter problem 1.13.