# Quantum Club

## *Entrance test 2023*

1. The base of a pyramid is a rectangle. Each of the side edges has length $l$ and makes angles $\alpha$ and $\beta$ with the base edges. Find the volume of the pyramid.

   Each side face of the pyramid is an isosceles triangle with base angle $\alpha$ or $\beta$. The bases of these triangles – of lengths $2l\cos\alpha$ and $2l\cos\beta$ – make up the pyramid base, whose area is hence $4l^2\cos\alpha\cos\beta$ To find the pyramid height, consider its cross-section with a vertical plane through the apex and middles of bases of length $2l\cos\alpha$. The cross-section is an isosceles triangle with the base $2l\cos\beta$ and sides $l\sin\alpha$. The height of this triangle (which is the height of the pyramid) is found from the Pythagorean theorem as $\sqrt{(l\cos\beta)^2-(l\sin\beta)^2}=l\sqrt{\sin^2\alpha-\cos^2\beta}$. Hence the pyramid volume is $\frac{4}{3}l^3\cos\alpha\cos\beta\sqrt{\sin^2\alpha-\cos^2\beta}$.

2. Find all solutions to the equation $\sqrt{1+\sin x}-\sqrt{1-\sin x}=1+\cos x$

   We can square both sides of equation, but we must keep in mind that the RHS of the original equation is nonnegative, so we must have $\sin x\ge 0$.

   $$\left(\sqrt{1+\sin x}-\sqrt{1-\sin x}\right)^2=(1+\cos x)^2$$

   $$2-2\sqrt{1-\sin^2 x}=(1+\cos x)^2$$

   $$2-2\cos x=(1+\cos x)^2 \text{ (if } \cos x>0) \quad \text{or} \quad 2+2\cos x=(1+\cos x)^2 \text{ (if } \cos x<0)$$

   $$\cos^2 x+4\cos x-1=0 \text{ (if } \cos x>0) \quad \text{or} \quad \cos^2 x-1=0 \text{ (if } \cos x<0)$$

   $$\cos x=-2\pm\sqrt{5} \text{ (only keep the + sign)} \quad \text{or} \quad \cos x=-1$$

   $$x=\pm\arccos\left(\sqrt{5}-2\right)+2\pi n \quad \text{or} \quad x=\pi+2\pi n$$

   Recalling that $\sin x\ge 0$, we drop the negative solution.

   Answer: $x=\arccos\left(\sqrt{5}-2\right)+2\pi n$ or $x=\pi+2\pi n$

3. Three trains are moving with a constant speed. After some time has elapsed, trains A and C together travelled twice as much distance as did train B, and trains B and C together – three times as much as A. Which train is the fastest?

   $$\begin{cases} v_A+v_C=2v_B \\ v_B+v_C=3v_A \end{cases}$$

   $$\begin{cases} 1+(v_C/v_A)=2(v_B/v_A) \\ (v_B/v_A)+(v_C/v_A)=3 \end{cases}$$

   Solving the simultaneous equations, we find $v_B/v_A=4/3, v_C/v_A=5/3$. Train C is the fastest.

4. Which term is the largest in the decomposition $\left(\sqrt{5}+\sqrt{2}\right)^{20}$?

   $$\left(\sqrt{5}+\sqrt{2}\right)^{20}=\sum_{k=0}^{20}C(20,k)\sqrt{5}^k\sqrt{2}^{20-k}=\sum_{k=0}^{20}\frac{20!}{k!(20-k)!}\sqrt{5}^k\sqrt{2}^{20-k}.$$

The ratio between the ($k$+1)th and $k$th terms is $\sqrt{\frac{5}{2}}\frac{20-k}{k+1}$. This is a function of k that increases for small k but decreases when k approaches 20. It decreases when

$$\sqrt{\frac{5}{2}}\frac{20-k}{k+1} > 1$$

$$\sqrt{\frac{5}{2}}(20-k) > k+1$$

$$k > \frac{20\sqrt{5/2}-1}{\sqrt{5/2}+1} \approx 11.9$$

Hence the term containing $\sqrt{5}^{12}$ is larger than both the $\sqrt{5}^{11}$ and $\sqrt{5}^{13}$ terms, so it is the largest.

5. Eight chess players A, B, C, D, E, F, G and H are playing in a tournament. Before the tournament, it is randomly decided which players play against each other in the first game. What is the probability that the pairs are AB, CD, EF and GH? The order of players in a pair does not matter, e.g. AB and BA is considered the same pair.
The number of possible pairings can be determined as follows. One can choose 7 possible opponents for A. Then 6 players remain unpaired. For the yet unpaired player with the "smallest" letter, one can choose 5 possible opponents. Then 4 players remain unpaired, for which 3 pairings are possible. Hence the number of possible pairings is $7 \cdot 5 \cdot 3 = 105$ and the probability of a specific pairing is 1/105.