Quantum Club

Entrance test 2021

Problems

1. Two objects are dropped from the same height at a 3-second interval with no initial velocity. How much time will elapse before they are 309 meters apart?

   $$\frac{g(t+3\text{s})^2}{2}-\frac{gt^2}{2}=309\text{m} \quad \Rightarrow \quad t=9 \text{ seconds}$$

2. What curve does the equation $x^2-ax+y^2-by=0$ (where $a>0, b>0$) represent?

   $\left(x-\frac{a}{2}\right)^2+\left(y-\frac{b}{2}\right)^2=\frac{a^2+b^2}{4}$. Circle with the center $(a/2, b/2)$ and radius $\sqrt{a^2+b^2}/2$

3. Two dice are tossed. What is the expectation and variance of the product of the two numbers they show?

   $$\langle n_1n_2\rangle=\langle n_1\rangle\langle n_2\rangle=49/4=12.25; \langle n_1^2n_2^2\rangle=\langle n_1^2\rangle\langle n_2^2\rangle=(91/6)^2=8281/36\approx 230;$$
   $$\langle \Delta(n_1n_2)^2\rangle=\langle n_1^2n_2^2\rangle-\langle n_1n_2\rangle^2=11515/144\approx 80$$

4. Solve the equation $\sin^4 x+\cos^4 x-\cos 2x=\frac{1}{2}$.

   $$1-2\sin^2 x\cos^2 x-\cos 2x=\frac{1}{2} \quad \Leftrightarrow \quad 1-\frac{\sin^2 2x}{2}-\cos 2x=\frac{1}{2}$$
   $$\Leftrightarrow \quad \frac{1}{2}+\frac{\cos^2 2x}{2}-\cos 2x=\frac{1}{2} \quad \Leftrightarrow \quad \cos 2x=0,\sqrt{2} \quad \Leftrightarrow \quad x=\frac{\pi}{4}+\frac{\pi}{2}m$$

5. Calculate $\int \cos x(1+\cos^2 x)\mathrm{d}x$

   $$\int \cos x(1+\cos^2 x)\mathrm{d}x=\int \cos x(2-\sin^2 x)\mathrm{d}x=\int (2-t^2)\mathrm{d}t=2t-\frac{t^3}{3}+C=2\sin x-\frac{\sin^3 x}{3}+C$$

   (use $t=\sin x; dt=\cos x dx$)

6. Simplify $\frac{\tan 607.5^\circ-\tan 22.5^\circ}{\tan 427.5^\circ+\tan 742.5^\circ}$

   $$\frac{\tan 607.5^\circ-\tan 22.5^\circ}{\tan 427.5^\circ+\tan 742.5^\circ}=\frac{\tan 67.5^\circ-\tan 22.5^\circ}{\tan 67.5^\circ+\tan 22.5^\circ}=\frac{\sin 45^\circ}{\sin 90^\circ}=\frac{1}{\sqrt{2}}$$

   (use $\tan x+\tan y=\frac{\sin x\cos y+\cos x\sin y}{\cos x\cos y}=\frac{\sin(x+y)}{\cos x\cos y}$)

7. Solve the equation $\log_2(x+1)=4-2x-x^2$. Hint: try to plot both sides of the equation.

   $x=1$

8. The four roots of the equation $x^4-ax^2+9=0$ make up an arithmetic progression. Find *a*.

   The roots are $\pm\sqrt{\frac{a\pm\sqrt{a^2-36}}{2}}$. Because they are symmetric around zero, we must have

$$\sqrt{\frac{a+\sqrt{a^2-36}}{2}} = 3\sqrt{\frac{a \pm \sqrt{a^2-36}}{2}}$$

$$\frac{a+\sqrt{a^2-36}}{2} = 9\frac{a-\sqrt{a^2-36}}{2}$$

$$10\sqrt{a^2-36} = 8a$$

$$100a^2 - 3600 = 64a^2$$

$$a = \pm 10$$

Only +10 is the correct answer because −10 will lead to a negative expression under the square root.