# Oxford Quantum Club

*Entrance test 19 December 2020*

## Problems

1. Find $\sin^4 x + \cos^4 x$, if $\sin x + \cos x = A$ is known.
2. Simplify the expression
   $$\cos^2\left(\frac{5}{8}\pi + \alpha\right) - \sin^2\left(\frac{15}{8}\pi + \alpha\right).$$
3. Calculate the integral $\int \sin^2 x \cos^2 x \, dx$
4. a) Prove that the definitions of complex conjugation
   - $a + ib \to a - ib$
   - $|z^*| = |z|; \arg(z^*) = -\arg(z)$

   are equivalent.

   b) Show that $\left(e^z\right)^* = e^{\left(z^*\right)}$.
5. For the standard deviation $\left\langle (\Delta x)^2 \right\rangle \equiv \left\langle \left( x - \langle x \rangle \right)^2 \right\rangle$ (where angular brackets denote averaging) of any random variable *x*, prove that $\left\langle (\Delta x)^2 \right\rangle = \left\langle x^2 \right\rangle - \left\langle x \right\rangle^2$.
6. A pile contains identically looking biased coins of one of three types, with the biases (probabilities of Heads) being 1/4, 1/2, or 3/4 and the probabilities of occurrence 1/6, 1/3 and 1/2, respectively. A coin is randomly chosen from a pile and tossed twice, producing Heads both times. Find the probability that the coin is of the third type.
7. Find the real and imaginary parts of the expression $\cos\left(\pi/2 + 2i\ln 2\right)$
8. Draw the trajectory of the tip of the electric field vector of an electromagnetic wave propagating along the *z* axis. The field vector is parallel to the $(x, y)$ plane and is described by the equation
   $$\mathbf{E}(z) = \begin{pmatrix} E_H(z) \\ E_V(z) \end{pmatrix} = \mathrm{Re}\left[\begin{pmatrix} E_{0H} \\ E_{0V} \end{pmatrix} e^{i(kz - \omega t)}\right],$$
   where the polarization state vector $\begin{pmatrix} E_{0H} \\ E_{0V} \end{pmatrix}$ is
   $$a)\begin{pmatrix} 1 \\ 0 \end{pmatrix}; b)\begin{pmatrix} 0 \\ 1 \end{pmatrix}; c)\begin{pmatrix} 5 \\ -3 \end{pmatrix}; d)\begin{pmatrix} 5i \\ -3 \end{pmatrix}; e)\begin{pmatrix} 5 \\ 3+4i \end{pmatrix}^*$$