Oxford Quantum Club

Entrance test 17 December 2020

Solutions

1. Find $\sin^4 x+\cos^4 x$, if $\sin x+\cos x=A$ is known.

   $$A^2=\sin^2 x+2\sin x\cos x+\cos^2 x=1+2\sin x\cos x \quad\Rightarrow\quad \sin x\cos x=\frac{A^2-1}{2}$$

   $$\sin^4 x+\cos^4 x=\left(\sin^2 x+\cos^2 x\right)^2-2\sin^2 x\cos^2 x=1-2\left(\frac{A^2-1}{2}\right)=-\frac{A^4}{2}+A^2+\frac{1}{2}$$

2. Simplify the expression $\cos^2\left(\frac{5}{8}\pi+\alpha\right)-\sin^2\left(\frac{15}{8}\pi+\alpha\right)$.

   $$\cos^2\left(\frac{5}{8}\pi+\alpha\right)-\sin^2\left(\frac{15}{8}\pi+\alpha\right)=\frac{1}{2}\left[1+\cos\left(\frac{5}{4}\pi+2\alpha\right)-1+\cos\left(\frac{15}{4}\pi+2\alpha\right)\right]$$

   $$=\cos\left(\frac{2\alpha+\frac{15}{4}\pi+2\alpha+\frac{5}{4}\pi}{2}\right)\cos\left(\frac{\frac{15}{4}\pi-\frac{5}{4}\pi}{2}\right)=\cos\left(2\alpha+\frac{\pi}{2}\right)\cos\left(\frac{5\pi}{4}\right)=\frac{\sin 2\alpha}{\sqrt{2}}$$

3. Calculate the integral $\int\sin^2 x\cos^2 x\,\mathrm{d}x$

   $$\int\sin^2 x\cos^2 x\,\mathrm{d}x=\frac{1}{4}\int\sin^2 2x\,\mathrm{d}x=\frac{1}{8}\int(1-\cos 4x)\,\mathrm{d}x=\frac{x}{8}-\frac{1}{32}\sin 4x+C$$

4. a) Prove that the definitions of complex conjugation
   - $a+ib\to a-ib$
   - $|z^*|=|z|;\arg(z^*)=-\arg(z)$

   are equivalent.

   b) Show that $\left(e^z\right)^*=e^{\left(z^*\right)}$.

   For $z=a+ib$, $\left(e^z\right)^*=e^a(\cos b+i\sin b)^*=e^a(\cos b-i\sin b)=e^{a-ib}=e^{\left(z^*\right)}$

5. For the variance $\left\langle(\Delta x)^2\right\rangle\equiv\left\langle\left(x-\langle x\rangle\right)^2\right\rangle$ (where angular brackets denote averaging) of any random variable *x*, prove that $\left\langle(\Delta x)^2\right\rangle=\left\langle x^2\right\rangle-\langle x\rangle^2$.

   $$\left\langle\left(x-\langle x\rangle\right)^2\right\rangle=\left\langle x^2-2x\langle x\rangle+\langle x\rangle^2\right\rangle=\left\langle x^2\right\rangle-2\langle x\rangle\langle x\rangle+\langle x\rangle^2=\left\langle x^2\right\rangle-\langle x\rangle^2$$

6. A pile contains identically looking biased coins of one of three types, with the biases (probabilities of Heads) being 1/4, 1/2, or 3/4 and the probabilities of occurrence 1/6, 1/3 and 1/2, respectively. A coin is randomly chosen from a pile and tossed twice, producing Heads both times. Find the probability that the coin is of the third type.

Suppose we perform the experiment *N* times, where *N* is large. Of these N times,

- we'll obtain the coin of the first type *N*/6 times, of which two Heads will occur $N/6 \times (1/4)^2 = N/96$ times;
- we'll obtain the coin of the second type *N*/3 times, of which two Heads will occur $N/3 \times (1/2)^2 = N/12$ times;
- we'll obtain the coin of the second type *N*/2 times, of which two Heads will occur $N/2 \times (3/4)^2 = 9N/32$ times.

Counting only the above events, the fraction of events in which the coin was of the third time is

$$\mathrm{pr}_3 = \frac{9N/32}{N/96 + N/12 + 9N/32} = \frac{3}{4}.$$

7. Find the real and imaginary parts of the expression $\cos(\pi/2 + 2i\ln 2)$

$$\cos(\pi/2 + 2i\ln 2) = \frac{1}{2}\left[e^{i\pi/2 - 2\ln 2} + e^{-i\pi/2 + 2\ln 2}\right] = \frac{1}{2}\left[ie^{-2\ln 2} - ie^{2\ln 2}\right] = \frac{i}{2}\left[\frac{1}{4} - 4\right] = -\frac{15i}{8}$$

8. Sketch the trajectory of the tip of the electric field vector of an electromagnetic wave propagating along the z axis. The field vector is parallel to the (x, y) plane and is described by the equation

$$\mathbf{E}(z) = \begin{pmatrix} E_H(z) \\ E_V(z) \end{pmatrix} = \mathrm{Re}\left[\begin{pmatrix} E_{0H} \\ E_{0V} \end{pmatrix} e^{i(kz - \omega t)}\right],$$

where the polarization state vector $\begin{pmatrix} E_{0H} \\ E_{0V} \end{pmatrix}$ is

$$a)\begin{pmatrix} 1 \\ 0 \end{pmatrix}; b)\begin{pmatrix} 0 \\ 1 \end{pmatrix}; c)\begin{pmatrix} 5 \\ -3 \end{pmatrix}; d)\begin{pmatrix} 5i \\ -3 \end{pmatrix}; e)\begin{pmatrix} 5 \\ 3+4i \end{pmatrix}^*$$