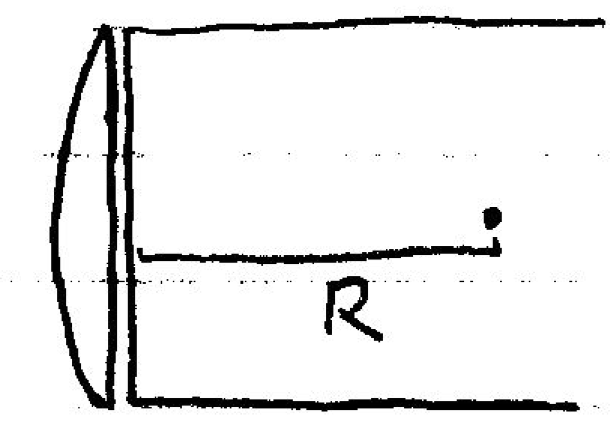
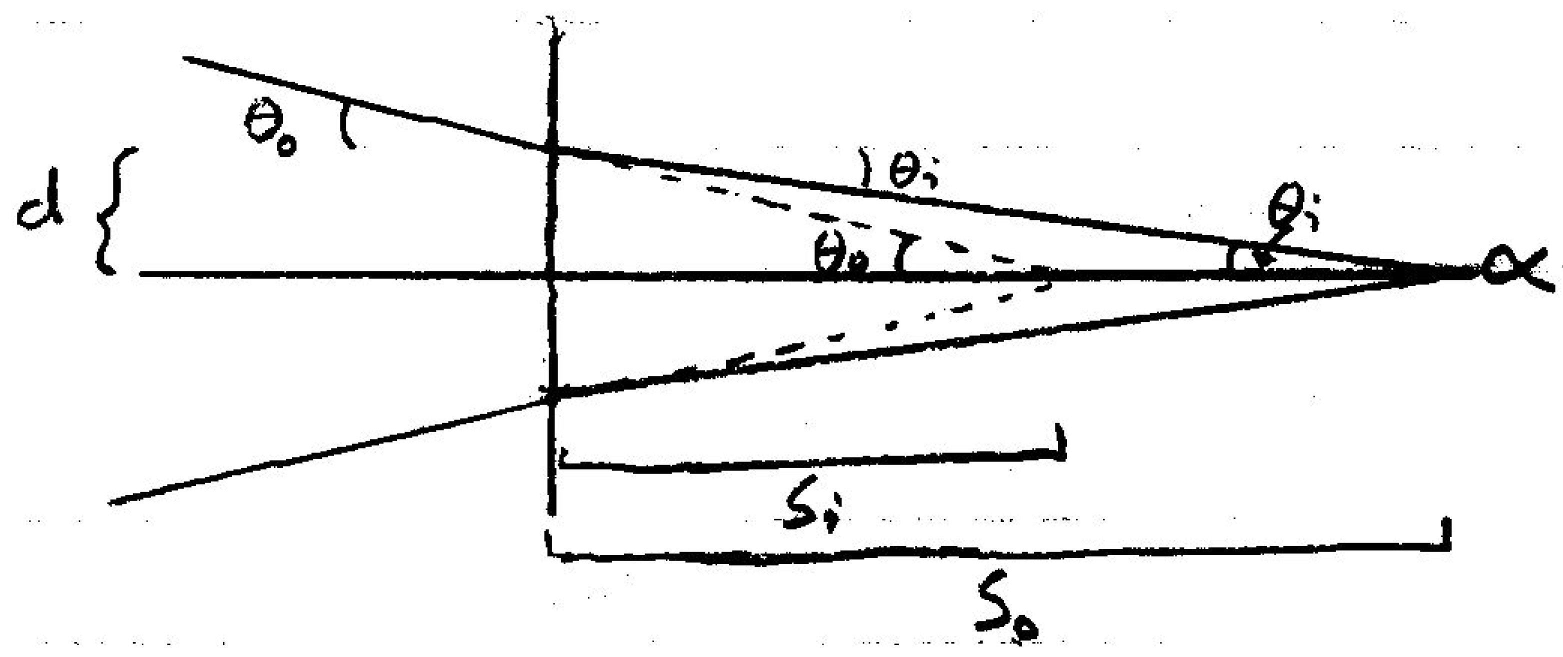
1.




We can treat the bowl as a thin lens of water followed by a block of water. So long as the glass is thin, it can be neglected.

Consider only the block of water:
It will create an image of the fish at $s = \frac{3}{4}R$




$n_a \sin\theta_o = n_w \sin\theta_i$

$\sin\theta_o = \frac{d}{s_i}$ $\sin\theta_i = \frac{d}{s_o}$

so $s_i = s_o / n_w = \frac{3}{4}R$

We then image the image at $\frac{3}{4}R$ through a thin lens.

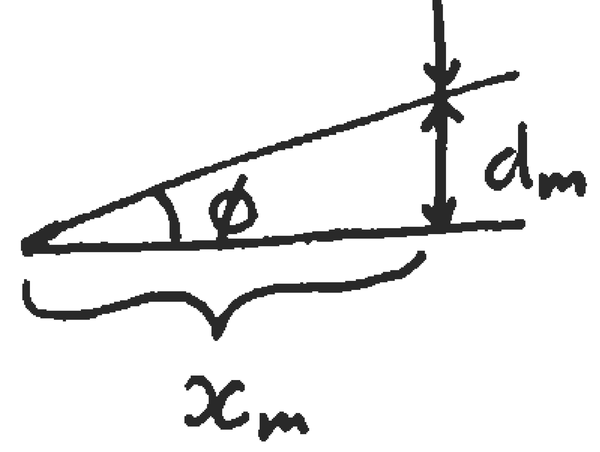
$$\frac{1}{s_o} + \frac{1}{s_i} = (n_w - 1)\left(\frac{1}{R}\right)$$ here $s_o \Rightarrow \frac{3}{4}R$

$$\Rightarrow \frac{1}{s_i} = \frac{1}{3}\frac{1}{R} - \frac{4}{3}\frac{1}{R} = \frac{-1}{R}$$ so $s_i = -R$

$$M = \frac{-s_i}{s_o} = \frac{4}{3}$$

So the fish will appear to be 6.7cm long.

2.




For a small angle $\phi$ we can neglect refraction off the glass-air interface. The condition for constructive interference is thus $d_m = (m + \frac{1}{2})\frac{\lambda}{2}$ for the $m^{th}$ bright band. Note that the $\frac{1}{2}$ added to $m$ compensates for the $\pi$ phase-shift upon reflection.

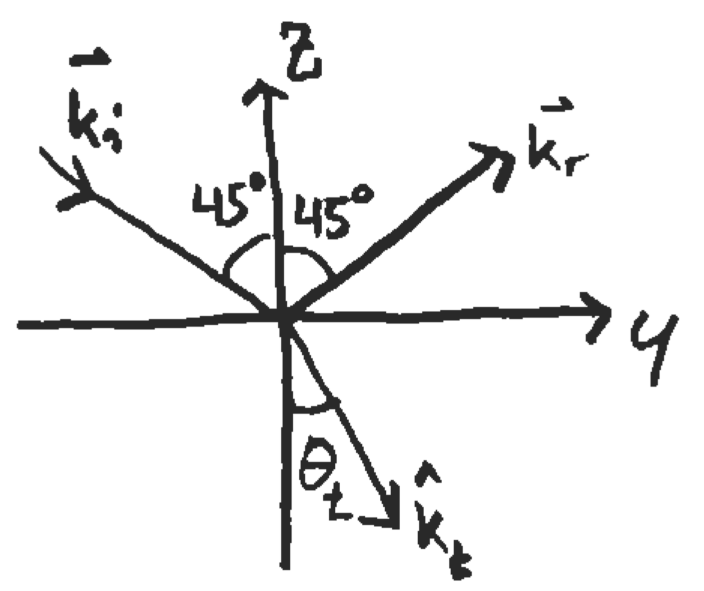
$\tan\phi = \frac{d_m}{x_m}$ , $\quad \phi = \pi/2 - \alpha/2$

$\tan(\phi) = \cot an(\alpha/2)$

So $x_m = (m + \frac{1}{2})\frac{\lambda}{2}\tan(89.95°)$

$x_m(\lambda) = 571(m + \frac{1}{2})\lambda$ for the $m^{th}$ bright fringe.

The interference pattern depends only on the distance from the prism apex (for a given $\lambda$). The pattern will thus be a series of alternating bright and dark parallel lines.

3. 


$\vec{E}_t = \vec{E}_{o_t} e^{i(\vec{k}_t \cdot \vec{r} - \omega t)}$ $\vec{k}_t \cdot \vec{r} = k_{t_y} y + k_{t_z} z$

$$k_{t_z} = k_t \cos\theta_t = \pm k_t \sqrt{1 - \sin^2\theta_t}$$

We apply Snell's Law: $k_{t_z} = \pm k_t \sqrt{1 - (n_i/n_t)^2 \sin^2\theta_i}$

So, $k_{t_z} = \pm k_t \sqrt{1 - \frac{1}{2}\left(\frac{n_i}{n_t}\right)^2}$

$= \pm i\beta k_t$ where $\beta = 0.226$ and $k_t = \frac{2\pi}{\lambda} = 9.94\times10^6\,m^{-1}$

$$\Rightarrow \boxed{\vec{E}_t = \vec{E}_{o_t} e^{-\beta k_t z} e^{i(k_{t_y} y - \omega t)}}$$

We reject the solution that gives us $e^{\beta k_t z}$ because it is unphysical (diverges at infinity)

$I = \frac{1}{2} c\epsilon_0 |E_0|^2$

$I_t = I_0 e^{-2\beta k_t z}$ $2\cdot\beta\cdot k_t = 4.5\times10^6\,m^{-1}$

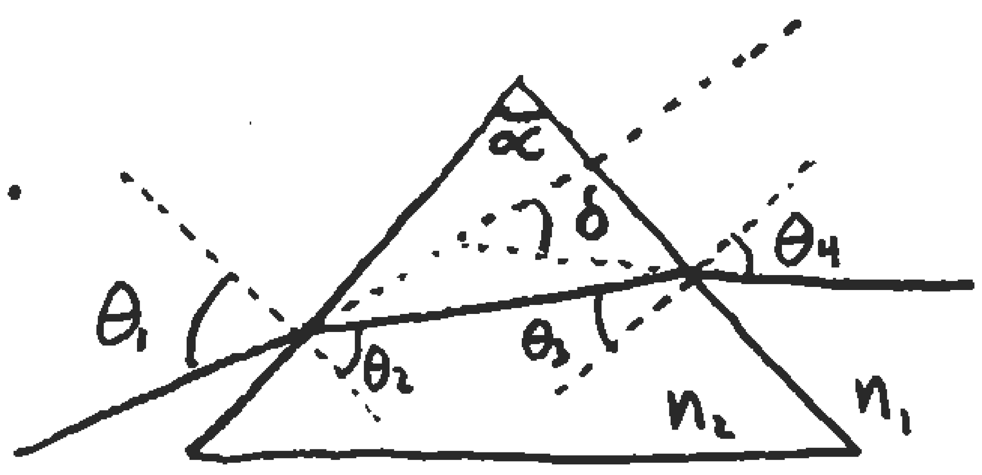
$= I_0 e^{-4.5\times10^6 m^{-1} z}$

For ½ intensity: $1/2 = e^{-4.5\times10^6 m^{-1} z}$

$\Rightarrow z_{1/2} = \dfrac{\ln(1/2)}{-4.5\times10^6\,m^{-1}}$

$$\boxed{z_{1/2} = 154\,nm}$$

4.




$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$
$$n_2 \sin\theta_3 = n_1 \sin\theta_4$$

$$\alpha + (90-\theta_2) + (90-\theta_3) = 180 \qquad \delta = \theta_1 + \theta_4 - \alpha$$
$$\Rightarrow \theta_3 = \alpha - \theta_2$$

$$\Rightarrow n_1 \sin\theta_4 = n_2 \sin(\alpha - \theta_2) \qquad \theta_4 = \sin^{-1}\left(\frac{n_2}{n_1}\sin(\alpha-\theta_2)\right) \quad \theta_1 = \sin^{-1}\left(\frac{n_2}{n_1}\sin(\theta_2)\right)$$

$$\delta = \sin^{-1}\left(\frac{n_2}{n_1}\sin(\alpha-\theta_2)\right) + \sin^{-1}\left(\frac{n_2}{n_1}\sin(\theta_2)\right) - \alpha$$

$$\frac{\partial\delta}{\partial\theta_2} = -\frac{n_2}{n_1}\frac{\cos^2\theta_3}{\sqrt{1-(n_2/n_1)^2\sin^2\theta_3}} + \frac{n_2}{n_1}\frac{\cos^2\theta_2}{\sqrt{1-(n_2/n_1)^2\sin^2\theta_2}} = 0$$

One solution is $\theta_2 = \theta_3$ (graphically, we see that it is the only valid solution).

5. We will use a unitary Fourier transform: $\mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$

a) $\mathcal{F}[\delta(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x) e^{-ikx} dx = \boxed{\frac{1}{\sqrt{2\pi}}}$

b) $\mathcal{F}[\delta(x-a/2)+\delta(x+a/2)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \, (\delta(x-a/2)+\delta(x+a/2)) e^{-ikx}$

$$= \frac{1}{\sqrt{2\pi}} \left( e^{-ika/2} + e^{ika/2} \right)$$

$$= \boxed{\sqrt{\frac{2}{\pi}} \cos\left(\tfrac{1}{2} ak\right)}$$

c) $\mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-b/2}^{b/2} e^{-ikx} dx$

$$= \frac{1}{\sqrt{2\pi}} \frac{i}{k} \left[ e^{-ikx} \right] \Big|_{x=-b/2}^{b/2} = \frac{1}{\sqrt{2\pi}} \frac{i}{k} \left[ e^{-ikb/2} - e^{ikb/2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{k} \sin\left(\tfrac{1}{2} bk\right)$$

$$= \boxed{\frac{b}{\sqrt{2\pi}} \operatorname{sinc}\left(\tfrac{1}{2} bk\right)}$$

d) We note that $f(x)$ is a convolution of the functions given in b) and c), we will call them $b(x)$ and $c(x)$ for consistency.

$$\mathcal{F}[f(x)] = \mathcal{F}[b(x) * c(x)] = \sqrt{2\pi}\, \mathcal{F}[b(x)] \mathcal{F}[c(x)]$$

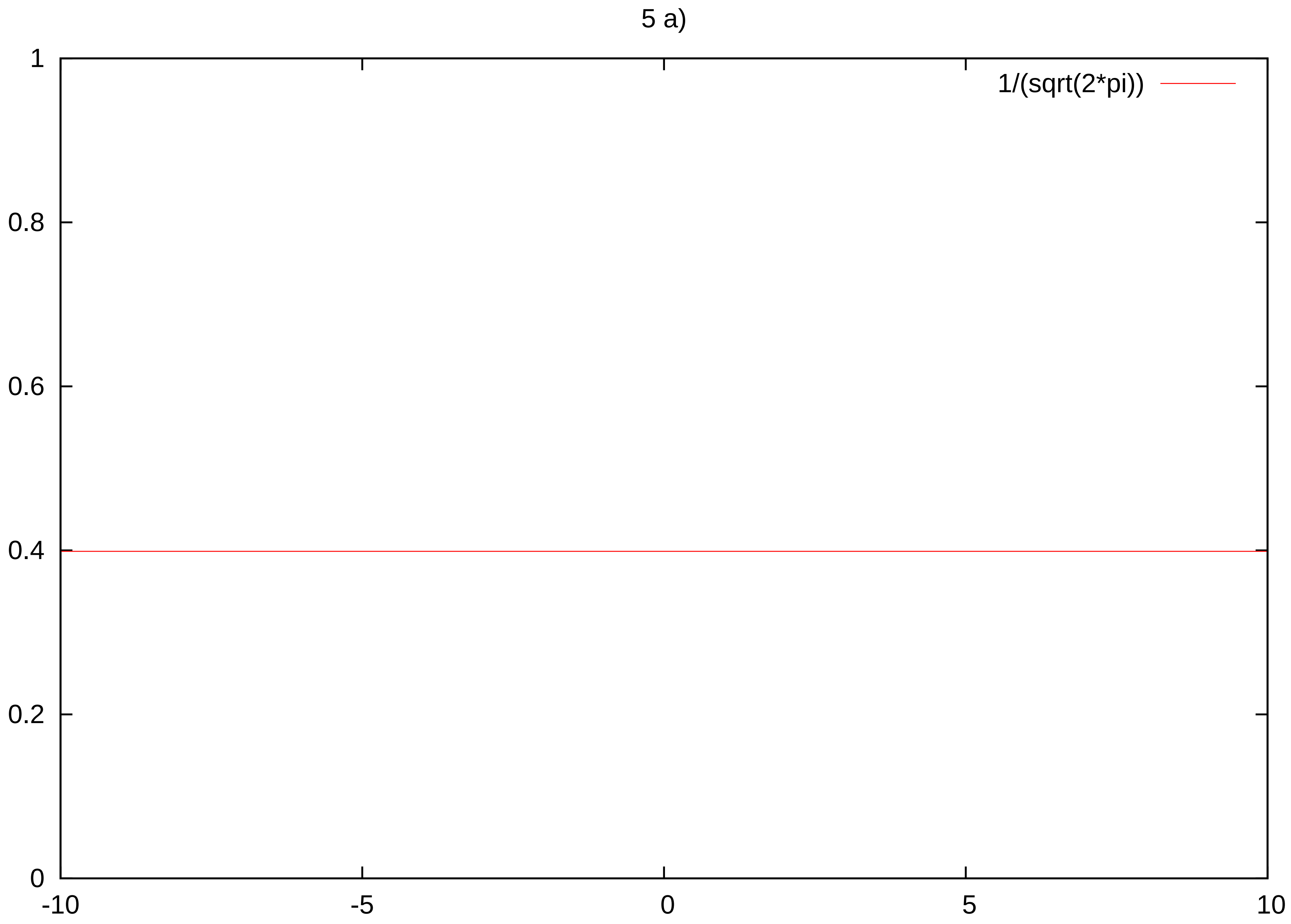
$$= \boxed{\sqrt{\frac{2}{\pi}} \cos\left(\tfrac{1}{2} ak\right) \operatorname{sinc}\left(\tfrac{1}{2} bk\right)}$$

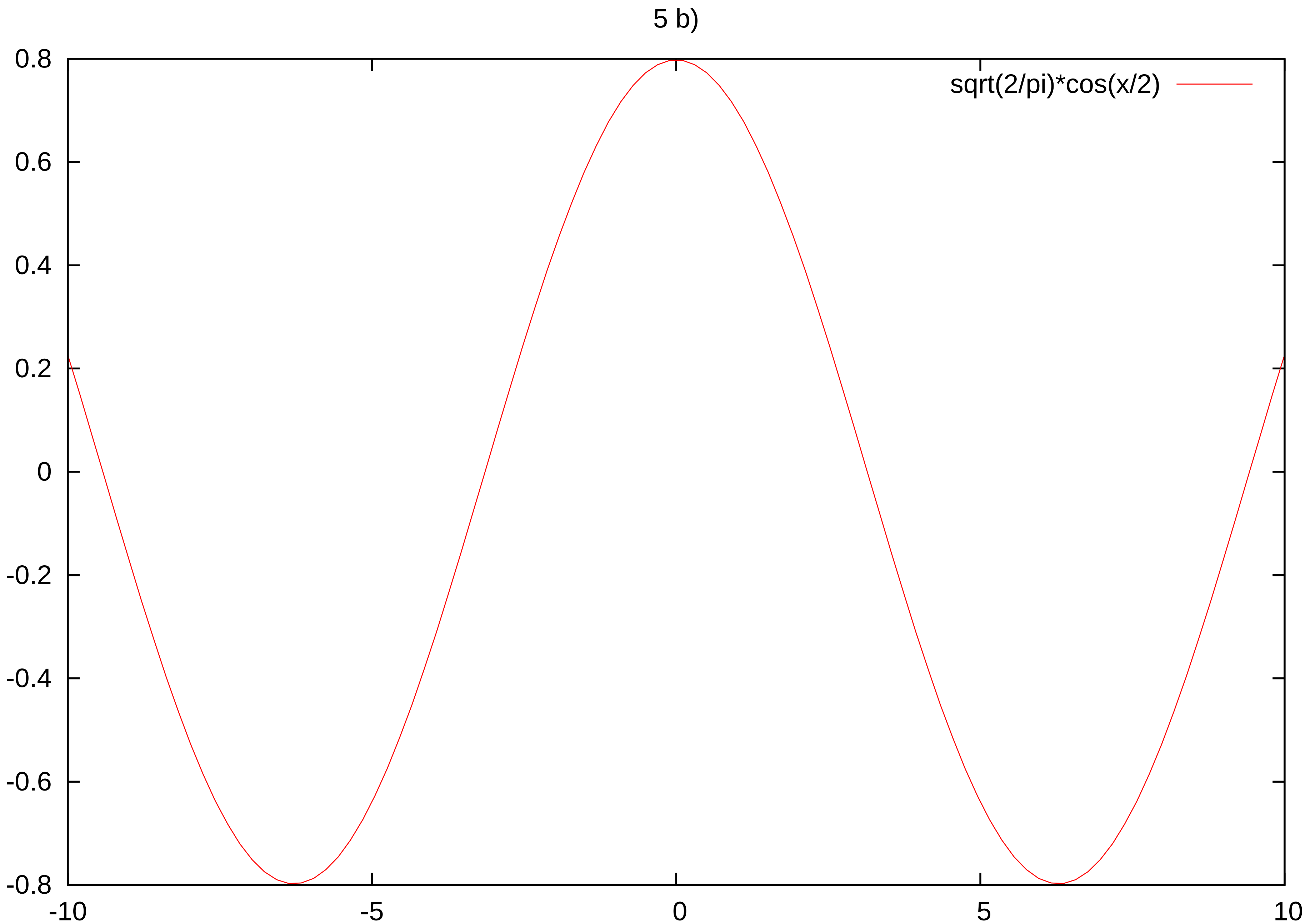
e) $\mathcal{F}\left[e^{-\left(\frac{x}{b}\right)^2}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x^2}{b^2} + ikx\right)} dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \, e^{-\left(\frac{x}{b} + \frac{ibk}{2}\right)^2 - \frac{b^2k^2}{4}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{b^2k^2}{4}} \int_{-\infty}^{\infty} e^{-\left(\frac{x}{b} + \frac{ibk}{2}\right)^2}$$

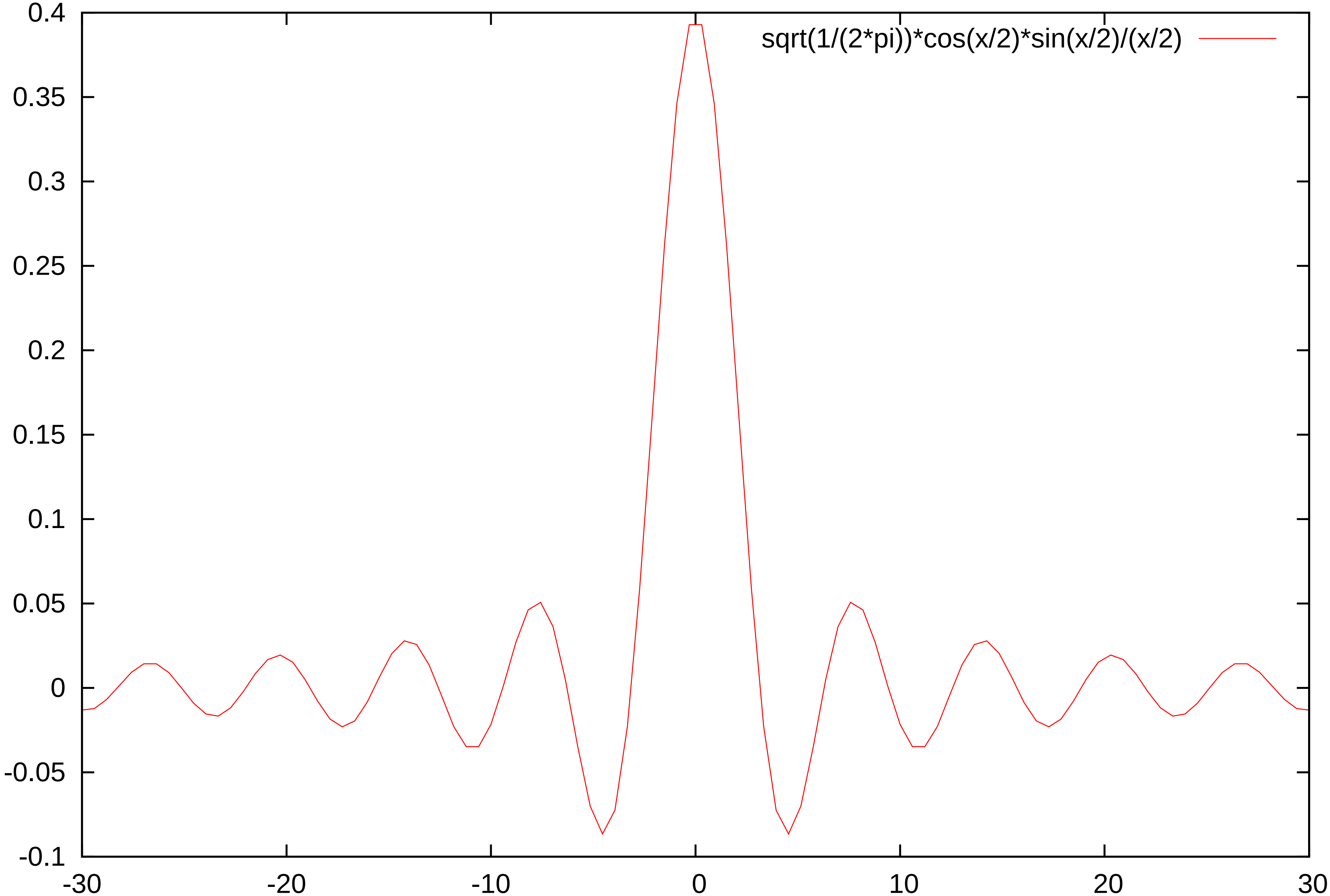
$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{b^2k^2}{4}} \frac{1}{b} \int_{-\infty}^{\infty} du \, e^{-u^2} \qquad ; u = \frac{x}{b} + \frac{ibk}{2}, \; du = \frac{1}{b} dx$$

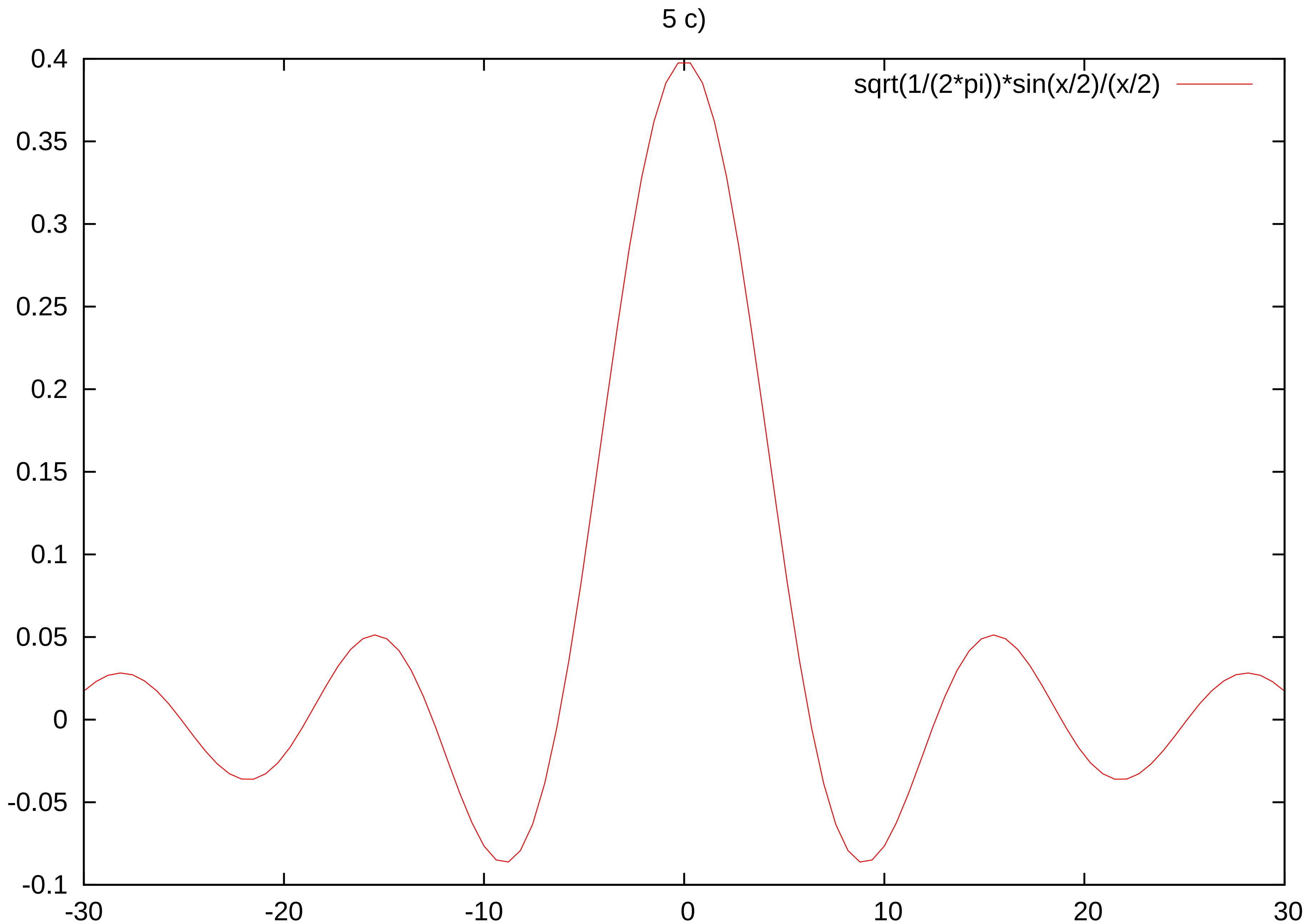
$$= \boxed{\frac{1}{\sqrt{2}\, b} e^{-\left(\frac{bk}{2}\right)^2}}$$


5 a)
1/(sqrt(2*pi))
1
0.8
0.6
0.4
0.2
0
-10
-5
0
5
10

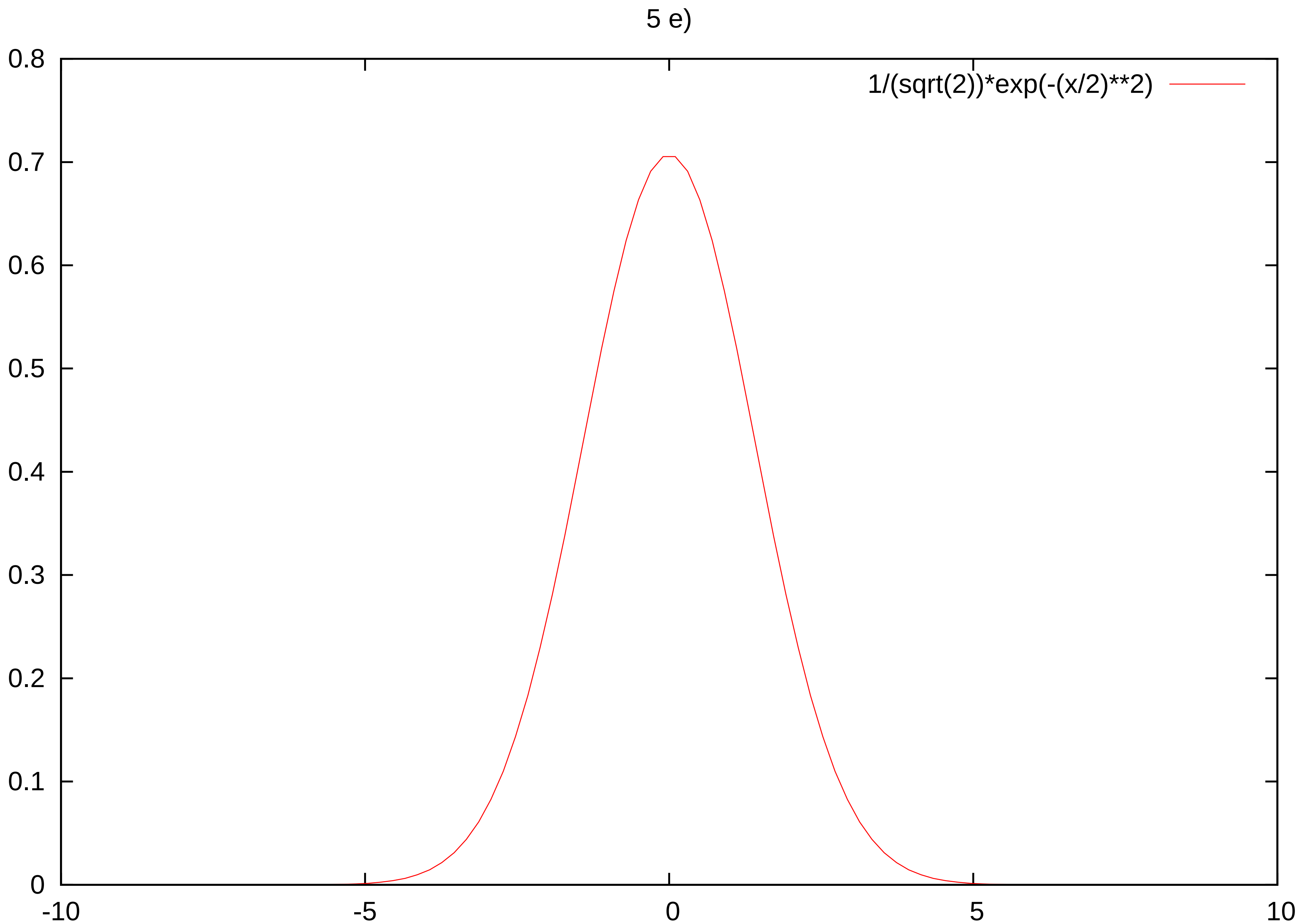

5 b)
sqrt(2/pi)*cos(x/2)
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-10
-5
0
5
10

5 d)


sqrt(1/(2*pi))*cos(x/2)*sin(x/2)/(x/2)
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-30
-20
-10
0
10
20
30



5 c)
sqrt(1/(2*pi))*sin(x/2)/(x/2)
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-30
-20
-10
0
10
20
30


5 e)
1/(sqrt(2))*exp(-(x/2)**2)
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-10
-5
0
5
10

7. a) $T(\lambda) = \begin{cases} 1 & (\lambda_0 - \frac{1}{2}\delta\lambda) < \lambda < (\lambda_0 + \frac{1}{2}\delta\lambda) \\ 0 & \text{otherwise} \end{cases}$

We can assume $T(\omega)$ of the form

$$T(\omega) \approx \begin{cases} 1 & (\omega_0 - \delta\omega) < \omega < (\omega_0 + \delta\omega) \\ 0 & \text{otherwise} \end{cases}$$

$$\omega_0 = 2\pi \frac{c}{\lambda_0}; \quad \delta\omega = \omega - \omega_0 = 2\pi c\left(\frac{1}{\lambda} - \frac{1}{\lambda_0}\right) = 2\pi c \frac{\delta\lambda}{\lambda\lambda_0} \approx 2\pi c \frac{\delta\lambda}{\lambda_0^2}$$

$$\delta\omega = 2\pi c \frac{\delta\lambda}{\lambda_0^2}$$

b) Consider a plane wave passing into the interferometer. The two components aquire a path length difference of $d$

$\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)} = \vec{E}_0 e^{i(k_x x - \omega t)}$ for a wave propagating along the $x$ axis.

For two waves, $\vec{E} = \vec{E}_{01} e^{i(k_{x_1} x_1 - \omega_1 t)} + \vec{E}_{02} e^{i(k_{x_2} x_2 - \omega_2 t)}$

$$= \frac{\vec{E}_0}{\sqrt{2}}\left(e^{i(k_x x_1 - \omega t)} + e^{i(k_x x_2 - \omega t)}\right)$$

$$= \frac{\vec{E}_0}{\sqrt{2}} e^{-i\omega t}\left(e^{ik_x x_1} + e^{ik_x x_2}\right)$$

We can ignore time-dependence and set $x_1 = 0$, $x_2 = x$

$$\langle \vec{E} \rangle_t = \frac{1}{\sqrt{2}} \vec{E}_0 (1 + e^{ik_x x})$$

Integrating over all wavelengths we get:

$$\langle \vec{E}_t \rangle = \frac{1}{\sqrt{2}} \int_0^\infty \vec{E}_0 (1 + e^{ik_x x})\, dk_x$$

$$= \frac{1}{\sqrt{2}} \int_0^\infty \vec{E}_0 (1 + e^{i\omega/c\, x}) \frac{1}{c}\, d\omega \qquad E_0(\omega) = E_0 T(\omega)$$

$$= \frac{\vec{E}_0}{\sqrt{2}} \int_{\omega_0 - \delta\omega}^{\omega_0 + \delta\omega} (1 + e^{i\omega/c\, x}) \frac{1}{c}\, d\omega$$

$$= \frac{\vec{E}_0}{\sqrt{2}} \left[ 2\delta\omega \frac{1}{c} + \left(\frac{-i}{x} e^{i\omega/c\, x}\right)\Bigg|_{\omega = \omega_0 - \delta\omega}^{\omega_0 + \delta\omega} \right]$$

$$= \frac{\vec{E}_0}{\sqrt{2}} \left[ 2\delta\omega \frac{1}{c} + \frac{1}{x} e^{i\omega_0/c\, x} \left(2\sin(\delta\omega/c\, x)\right) \right]$$

$$\langle E_t \rangle = \sqrt{2}\, E_0 \frac{\delta\omega}{c}\left[1 + \mathrm{sinc}(\delta\omega/c\, x)\, e^{i\omega_0/c\, x}\right]$$

$$I = E_0^2 \frac{\epsilon_0}{c} \delta\omega^2 \left[1 + \mathrm{sinc}(\tfrac{\delta\omega}{c}x)\, \mathrm{Re}[e^{i\omega_0/c\, x}]\right]^2$$

$$I \propto \delta\omega^2 \left[1 + \mathrm{sinc}(\tfrac{\delta\omega}{c}x)\cos(\tfrac{\omega_0}{c}x)\right]^2$$

$$V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}}$$

$\cos(\frac{\omega_0}{c}x)$ oscillates much more quickly than $\mathrm{sinc}(\frac{\delta\omega}{c}x)$ so we consider the visibility to be the difference between $I(\cos(\frac{\omega_0}{c}x))_{max}$ and $I(\cos(\frac{\omega_0}{c}x))_{min}$ varying slowly with $\mathrm{sinc}(\frac{\delta\omega}{c}x)$

$$V(x) = \frac{(1 + \mathrm{sinc}(\frac{\delta\omega}{c}x))^2 - (1 - \mathrm{sinc}(\frac{\delta\omega}{c}x))^2}{(1 + \mathrm{sinc}(\frac{\delta\omega}{c}x))^2 + (1 - \mathrm{sinc}(\frac{\delta\omega}{c}x))^2}$$

c) $V(x)$ drops to zero when $\sin(\frac{\delta\omega}{c}x) = 0 \Rightarrow x = \frac{2\pi c}{\delta\omega}$ is the coherence length.

The coherence time $\tau_c = \frac{x}{c} = \frac{2\pi}{\delta\omega} = \frac{\lambda_0^2}{\delta\lambda}\frac{1}{c}$

$$\tau_c = 21\,\mathrm{ps}$$

$\tau_c \times \delta\omega = 2\pi \sim 1$ (Same order of magnitude)