University of Calgary
Winter semester 2017

PHYS 443: Quantum Mechanics I

Homework assignment 5

Due March 28, 2017

Problem 5.1. For the operators (i) $\hat{A} = e^{\frac{i}{\hbar}a\hat{p}}$ and (ii) $\hat{A} = e^{-\frac{\hat{p}^2 d^2}{4\hbar^2}}$, determine

a) the matrix element $\left\langle x' \middle| \hat{A} \middle| x'' \right\rangle$;

b) the wavefunction of state $\hat{A}\left|\psi\right\rangle$ if the wavefunction $\psi(x)$ of state $\left|\psi\right\rangle$ is known.

Problem 5.2. Recalling that $\mathrm{pr}(x) = \psi(x)\psi^*(x)$, derive the continuity equation:

$$\frac{d\mathrm{pr}(x)}{dt} = -\frac{dj}{dx}, \tag{1}$$

where

$$j = -i\frac{\hbar}{2m}\left(\psi^*\frac{d\psi}{dx} - \psi\frac{d\psi^*}{dx}\right) \tag{2}$$

is the probability density current.

Problem 5.3. Consider a photon that has both the polarization and spatial degrees of freedom, which are treated as a tensor product Hilbert space. Initially, the photon is polarized at angle θ to horizontal and has a Gaussian wavefunction, so its state can be written as $\left|\Psi(0)\right\rangle = \left|\theta\right\rangle \otimes \left|G_{b,0}\right\rangle$ with

$$\left|G_{b,x_0}\right\rangle = \frac{1}{\pi^{1/4}\sqrt{b}} \int\limits_{-\infty}^{+\infty} e^{-(x-x_0)^2/2b^2} \left|x\right\rangle \mathrm{d}x.$$

The photon passes through a birefringent element that implements polarization-dependent spatial displacement that corresponds to the evolution under the Hamiltonian

$$\hat{H} = \hbar a \hat{\sigma}_x \otimes \hat{p}$$

for time t.

a) Find the evolution $\left|\Psi(t)\right\rangle$ of the photon's state.
Hint: write the polarization in the diagonal basis and use the result of Problem 5.1 above.

b) After this evolution, the photon passes through a PBS that transmits horizontal polarization and reflects vertical. Find the spatial wavefunction of the reflected photon.

c) Find the mean position of the reflected photon.
Hint: don't forget to re-normalize.

Problem 5.4. Solve Ex. 3.37 from the lecture notes in the momentum basis and verify consistency with the position-basis solution through the following steps.

a) Calculate the matrix element $\left\langle p \middle| \hat{V} \middle| p' \right\rangle$ of the potential in the momentum basis and the momentum-basis wavefunction $\left\langle p \middle| \hat{V} \middle| \psi \right\rangle$ of vector $\hat{V}\,|\psi\rangle$ for an arbitrary state $|\psi\rangle$.

   Hint: your answer should contain $\int\limits_{-\infty}^{+\infty} \psi(p)\mathrm{d}p$.

b) Write the time-independent Schrödinger equation in the momentum basis and solve for $\psi(p)$. Calculate the energy value for which your solution is self-consistent.

   Hint: $\int\limits_{-\infty}^{+\infty} \frac{1}{1+x^2}\mathrm{d}x = \pi$.

c) Calculate the Fourier transform of the position-basis wavefunction determined in Ex. 3.37 and check that it is consistent with the momentum-basis wavefunction you found in part (b). You need not worry about the normalization.

Problem 5.5. Find the transcendental equation for the energies corresponding to the solutions of the time-independent Schrödinger equation in the potential

$$V(x) = \begin{cases} -W_0\delta(x), & |x| \leq a; \\ +\infty, & |x| > a \end{cases}$$

(see below). For which W_0, a does at least one solution exist?