LECTURE 5:

VECTOR GEOMETRY : REPRESENTATION OF PLANES

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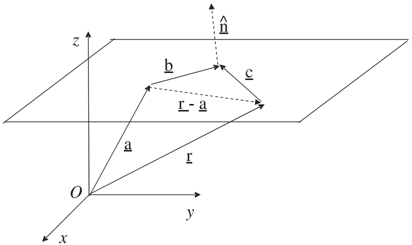
Outline: 5. MORE ON VECTOR GEOMETRY

5.1 Vector representation of planes

- Vector $\underline{\mathbf{a}}$ is any position vector to the plane. Vectors $\underline{\mathbf{b}}$ and $\underline{\mathbf{c}}$ are any vectors in the plane (but not parallel to each other). $\underline{\mathbf{r}}$ is a position vector to a general point on the plane.



- The equation of the plane can then be written by:

  $\underline{\mathbf{r}} = \underline{\mathbf{a}} + \lambda \underline{\mathbf{b}} + \mu \underline{\mathbf{c}}$

  where λ and μ take all values to give all positions on the plane.

- Conversely, it should be obvious that a vector equation for the plane can be more simply written:

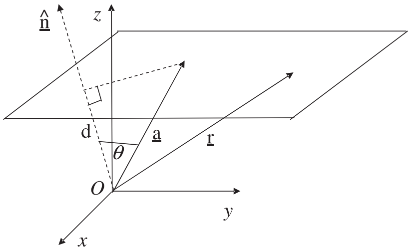
$$(\underline{\mathbf{r}} - \underline{\mathbf{a}}).\underline{\hat{\mathbf{n}}} = 0$$

where $\underline{\hat{\mathbf{n}}}$ $(= \frac{\underline{\mathbf{b}} \times \underline{\mathbf{c}}}{|\underline{\mathbf{b}} \times \underline{\mathbf{c}}|})$ is the unit vector perpendicular to the plane.

5.1.1 Plane from vector to Cartesian form

- $(\underline{\mathbf{r}} - \underline{\mathbf{a}}).\underline{\hat{\mathbf{n}}} = 0$ gives $\underline{\mathbf{r}}.\underline{\hat{\mathbf{n}}} = \underline{\mathbf{a}}.\underline{\hat{\mathbf{n}}}$

- Note that $d = a\cos\theta = \underline{\mathbf{a}}.\underline{\hat{\mathbf{n}}}$ is the perpendicular distance of the plane to the origin.
- Also we write $\underline{\hat{\mathbf{n}}} = l\underline{\mathbf{i}} + m\underline{\mathbf{j}} + n\underline{\mathbf{k}}$. where (l, m, n) are defined as the *direction cosines* of the normal to the plane.



- Finally we write the general vector $\underline{\mathbf{r}}$ as (x, y, z)
- This gives the plane in Cartesian representation as

$$\underline{\mathbf{r}}.\underline{\hat{\mathbf{n}}} = lx + my + nz = d$$

5.1.2 From components back to vector form

- If a plane is represented by the coordinate equation $n_x x + n_y y + n_z z = \lambda$ (here n_x, n_y, n_z, λ are *any* values) Then a normal vector to the plane is simply $\underline{\mathbf{n}} = (n_x, n_y, n_z)$.

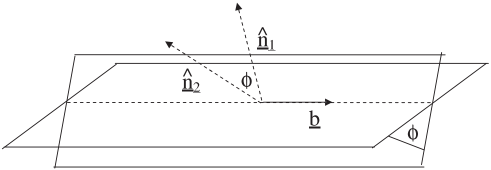
Example

- Coordinate equation $5x + 3y + z = 6$: the normal vector to the plane is $(5, 3, 1)$ and $\underline{\hat{\mathbf{n}}} = \frac{(5,3,1)}{\sqrt{(25+9+1)}} = (5, 3, 1)/\sqrt{35}$.
- So vector equation of plane is $\underline{\mathbf{r}}.\underline{\hat{\mathbf{n}}} = d$ where $d = 6/\sqrt{35}$ is the perpendicular distance.

5.2 Two intersecting planes

- The angle ϕ between the planes is the angle between the two normal vectors of the planes:
  cos ϕ = $\underline{\hat{\mathbf{n}}}_{\mathbf{1}}.\underline{\hat{\mathbf{n}}}_{\mathbf{2}}$

- The planes are parallel if cos ϕ = 1



- The direction of the line of intersection of the two planes:

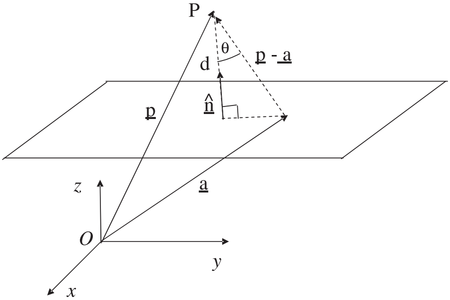
$$\underline{\hat{\mathbf{b}}}_{\textit{Line of intersection}} = \underline{\hat{\mathbf{n}}}_{\mathbf{1}} \times \underline{\hat{\mathbf{n}}}_{\mathbf{2}}$$

  i.e. parallel to both planes and perpendicular to both normals.

5.3 Minimum distance from a point to a plane

- Find the minimum distance, *d*, from point P with position vector $\underline{\mathbf{p}}$, to the plane defined by $(\underline{\mathbf{r}} - \underline{\mathbf{a}}).\underline{\hat{\mathbf{n}}} = 0$

- Consider vector $(\underline{\mathbf{p}} - \underline{\mathbf{a}})$ which is a vector from the plane to the point P



- The component of $(\underline{\mathbf{p}} - \underline{\mathbf{a}})$ normal to the plane is equal to the minimum distance of P to the plane.

i.e. $d = (\underline{\mathbf{p}} - \underline{\mathbf{a}}) \,.\, \underline{\hat{\mathbf{n}}}$

(sign depends on which side of plane the point is situated).

Example

- Three points lie on a plane: $(2,1,2),(-1,-1,-1)$ and $(4,1,2)$. Find the shortest distance of this plane from the point $(1,1,1)$.

Solution:

- $\underline{\mathbf{p}} = (1,1,1), \quad \underline{\mathbf{a}} = (2,1,2), \quad (\underline{\mathbf{p}} - \underline{\mathbf{a}}) = (-1,0,-1)$
- Construct two lines in the plane:

  $\underline{\mathbf{b}} = (2,1,2) - (-1,-1,-1) = (3,2,3)$
  $\underline{\mathbf{c}} = (2,1,2) - (4,1,2) = (-2,0,0)$
- A normal to the plane is:

$$\underline{\mathbf{n}} = \underline{\mathbf{b}} \times \underline{\mathbf{c}} = \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ 3 & 2 & 3 \\ -2 & 0 & 0 \end{vmatrix} \tag{1}$$

  giving $\underline{\mathbf{n}} = (0, -3 \times 2, 2 \times 2)$, $\quad \underline{\hat{\mathbf{n}}} = (0,-6,4)/\sqrt{(6^2+4^2)}$
- Therefore $d = (\underline{\mathbf{p}} - \underline{\mathbf{a}}) \,.\, \underline{\hat{\mathbf{n}}} = (-1,0,-1) \,.\, (0,-6,4)/\sqrt{(52)}$

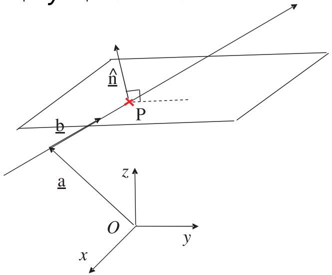
  $d = -4/\sqrt{(52)}$; $\quad |d| = 4/\sqrt{(52)}$

  (the minus sign specifies which side of the plane P is located).

5.4 Intersection of a line with a plane

- Example: A line is given by $\underline{\mathbf{r}} = \underline{\mathbf{a}} + \lambda\underline{\mathbf{b}}$ where $\underline{\mathbf{a}} = \underline{\mathbf{i}} + 2\underline{\mathbf{j}} + 3\underline{\mathbf{k}}$ and $\underline{\mathbf{b}} = 4\underline{\mathbf{i}} + 5\underline{\mathbf{j}} + 6\underline{\mathbf{k}}$. Find the coordinates of the point at which the line intersects the plane $2x + y + 3z = 6$.

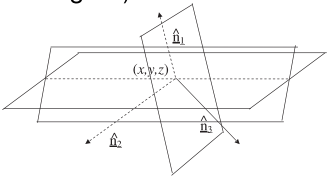
- A normal vector to the plane is $\underline{\mathbf{n}} = (2, 1, 3)$.
- First check that the line and plane are not parallel (i.e. $\underline{\mathbf{b}}$ and $\underline{\mathbf{n}}$ are not at 90°):
  $\underline{\mathbf{b}} \,.\, \underline{\mathbf{n}} = (4, 5, 6) \,.\, (2, 1, 3) =$
  $8 + 5 + 18 = 31 \neq 0$
- Therefore the line crosses the plane.



- To get the intersection point, substitute $\underline{\mathbf{r}} = \underline{\mathbf{a}} + \lambda\underline{\mathbf{b}}$ into equation of plane
  $\Rightarrow \quad (x, y, z) = (a_x + \lambda b_x, a_y + \lambda b_y, a_z + \lambda b_z)$ into $2x + y + 3z = 6$.
  $\Rightarrow \quad 2 \times (1 + 4\lambda) + (2 + 5\lambda) + 3 \times (3 + 6\lambda) = 6$
  $\Rightarrow \quad 13 + 31\lambda = 6 \quad \Rightarrow \quad \lambda = -7/31.$
- Substituting λ into the equation of the line
  $x = 1 - (7/31) \times 4 = (3/31)$
  $y = 2 - (7/31) \times 5 = (27/31)$
  $z = 3 - (7/31) \times 6 = (51/31)$

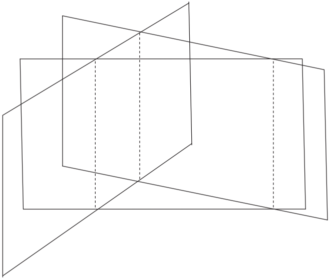
5.5 Intersection of three planes

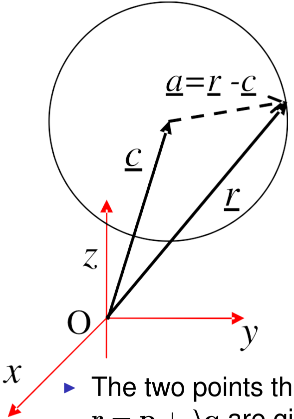
- Three planes intersect at a single point provided any two are not parallel ($\hat{\underline{\mathbf{n}}}_1 \times \hat{\underline{\mathbf{n}}}_2 \neq 0$, $\hat{\underline{\mathbf{n}}}_1 \times \hat{\underline{\mathbf{n}}}_3 \neq 0$, $\hat{\underline{\mathbf{n}}}_2 \times \hat{\underline{\mathbf{n}}}_3 \neq 0$)
  AND provided that any one of the planes is not parallel to the line of intersection of the other two (bottom figure).



- The sufficient condition for intersection is that the scalar triple product $\hat{\underline{\mathbf{n}}}_1 . (\hat{\underline{\mathbf{n}}}_2 \times \hat{\underline{\mathbf{n}}}_3) \neq 0$.
- Assuming a single solution, to get the point of intersection (x, y, z), easiest just to solve the equations:

$$\begin{aligned} l_1 x &+ m_1 y + n_1 z = d_1 \\ l_2 x &+ m_2 y + n_2 z = d_2 \\ l_3 x &+ m_3 y + n_3 z = d_3 \end{aligned}$$





5.6 Vector representation of a sphere

$$|\underline{\mathbf{r}} - \underline{\mathbf{c}}|^2 = a^2$$

alternatively

$$r^2 - 2\underline{\mathbf{r}} \cdot \underline{\mathbf{c}} + c^2 = a^2$$

- $\underline{\mathbf{c}}$ is the position vector to the centre of the sphere
- $a = |\underline{\mathbf{a}}|$ is the sphere radius (scalar)

- The two points that are the intersection of the sphere with a line $\underline{\mathbf{r}} = \underline{\mathbf{p}} + \lambda\underline{\mathbf{q}}$ are given by solving the quadratic for λ :

$$(\underline{\mathbf{p}} + \lambda\underline{\mathbf{q}} - \underline{\mathbf{c}}) \cdot (\underline{\mathbf{p}} + \lambda\underline{\mathbf{q}} - \underline{\mathbf{c}}) = a^2$$

- The radius ρ of the circle that is the intersection of the sphere with a plane $\underline{\hat{\mathbf{n}}} \cdot \underline{\mathbf{r}} = d$ is given by

$$\rho = \sqrt{a^2 - (d - \underline{\mathbf{c}} \cdot \underline{\hat{\mathbf{n}}})^2}$$

(See Riley, Hobson & Bence for proof.)