THE FERMI GAS AT LOW TEMPERATURE

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We discuss basic properties of a gas of fermions at low temperature. We shall get some intuition about what the particles are doing by examining the Fermi-Dirac distribution function. We shall find the energy of the gas, and the pressure at zero temperature, finding that it is surprisingly high (not zero as one might guess). We shall also estimate the heat capacity at low temperature, and paramagnetic properties for electrons.

1 Fermi energy

Choosing the zero of energy. The physics of a gas at low temperature depends on the gaps between the energy levels, but not on the value of the ground state energy itself. This is illustrated by the fact that the energy levels ϵ_i do not appear on their own in the partition function, but in company with the chemical potential, in the term $\beta(\mu - \epsilon_i)$. If we shift the zero of energy, both μ and ϵ_0 will move and $\mathcal{Z}_i$ is unaffected. It is customary, and convenient, to choose as zero energy the single-particle ground state energy. Thus if single-particle state number 1 is the ground state (which is the convention we shall adopt, so that the occupation numbers are $n_1, n_2, \cdots$) then $\epsilon_1 = 0$.

The ground state energy of the entire gas (of many fermions) is not 0 because multiple fermions cannot all occupy the single-particle ground state.

Now consider the Fermi-Dirac distribution function, plotted in Fig. 1 for four example temperatures. At $T = 0$ the prediction is that states of low energy, $\epsilon < \mu$, are fully occupied ($\langle n \rangle = 1$) and states of energy above μ have zero occupation ($\langle n \rangle = 0$). Now μ is itself a function of temperature, but since its value at $T = 0$ has this special significance in the distribution, we give it a name: it is called the *Fermi energy*:

$$\epsilon_{\mathrm{F}} \equiv \mu(T = 0). \tag{1}$$

In the case of a gas in a box we can calculate the Fermi energy as follows. We recall the idea of states of motion characterised by wave vectors $\mathbf{k}$, with one state per region of size $(2\pi/L)^d$ in k-space, when

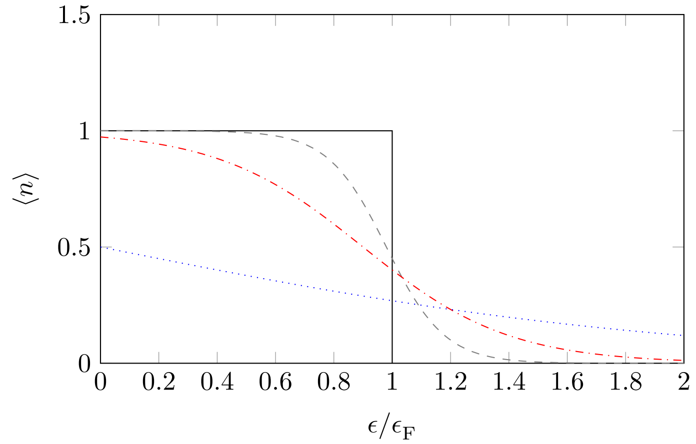

Figure 1: The Fermi-Dirac distribution function at 4 example temperatures. We plot the mean occupation of a state of energy ϵ, as a function of $\epsilon/\epsilon_\mathrm{F}$ where ϵ_F is the Fermi energy (see text). Full curve: $T = 0$; gray dashed: $k_\mathrm{B}T = 0.1\,\epsilon_\mathrm{F}$; red dash-dot: $k_\mathrm{B}T = 0.25\,\epsilon_\mathrm{F}$; blue dots: $k_\mathrm{B}T = \epsilon_\mathrm{F}$. In order to plot the function at any given T, one must allow for the fact that μ falls below ϵ_F as T increases from zero (see Fig. 3). One can locate μ on the graph of $\langle n(\epsilon)\rangle$ by observing that $\langle n \rangle = 0.5$ when $\epsilon = \mu$.

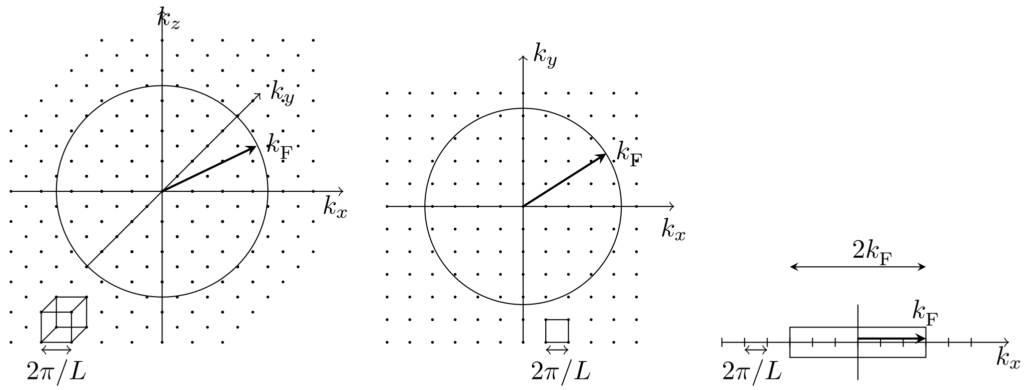

Figure 2: Obtaining the Fermi k-vector, in 3,2 and 1 dimensions. k_F is the Fermi wave-vector when the number of states in the sphere, circle, rectangle, respectively, after being multiplied by the spin-state factor $(2s + 1)$, is equal to the number of particles, N.

Density of states for free particles in a box (without spin)

$$g(\epsilon) = \frac{L}{2\pi\hbar}\,(2m)^{1/2}\,\epsilon^{-1/2} \qquad \text{1d} \tag{5}$$

$$g(\epsilon) = \frac{Am}{2\pi\hbar^2} \qquad \text{2d} \tag{6}$$

$$g(\epsilon) = \frac{V}{4\pi^2\hbar^3}(2m)^{3/2}\epsilon^{1/2} \qquad \text{3d} \tag{7}$$

The above can also be presented as follows:

$$g(\epsilon) = D\epsilon^q \qquad \begin{array}{ccc} \text{dimensions} & D & q \\ 1 & L(2m)^{1/2}/(2\pi\hbar) & -1/2 \\ 2 & A(2m)\,/\,(4\pi\hbar^2) & 0 \\ 3 & V(2m)^{3/2}/(4\pi^2\hbar^3) & 1/2 \end{array} \tag{8}$$

In the case of spin-half particles these values of D must be multiplied by $(2s+1) = 2$ to get the total density of states.

the gas is in a d-dimensional box of side L (see Fig. 2), and we have adopted periodic boundary conditions. We note that at zero temperature all states of energy up to $\epsilon_{\rm F}$ are occupied, and none above that. Writing $\epsilon_{\rm F} = (\hbar k_{\rm F})^2/2m$ we deduce that all states up to wave-vector size $k_{\rm F}$ are occupied, where $k_{\rm F}$ is to be obtained. In 3 dimensions the occupied states fill a sphere of radius $k_{\rm F}$ and there must be N of them, so we find

$$(2s+1)\frac{(4/3)\pi k_{\rm F}^3}{(2\pi/L)^3} = N \tag{2}$$

where s is the spin of the particles making up the gas (for fermions we are usually concerned with $s = 1/2$). Similar calculations in 2d and 1d give

$$(2s+1)\frac{\pi k_{\rm F}^2}{(2\pi/L)^2} = N \qquad \text{2d}$$

$$(2s+1)\frac{2k_{\rm F}}{(2\pi/L)} = N \qquad \text{1d} \tag{3}$$

and therefore, for spin $s = 1/2$, the three cases give

$$k_{\rm F}^3 = \frac{N}{V}3\pi^2, \qquad k_{\rm F}^2 = \frac{N}{A}2\pi, \qquad k_{\rm F} = \frac{N}{L}\frac{\pi}{2}, \qquad \text{(3d, 2d, 1d.)} \tag{4}$$

From these results $\epsilon_{\rm F}$ can be found immediately, using $\epsilon_{\rm F} = (\hbar k_{\rm F})^2/2m$.

It will be useful in the following to observe that the Fermi energy can also be found via the density of states as a function of energy. We use the same basic idea, that the occupation numbers must sum to N:

$$\int_0^\infty g(\epsilon)\bar{n}(\epsilon)\mathrm{d}\epsilon = \mathrm{N} \tag{9}$$

where $g(\epsilon)$ is the density of states function, we have approximated the sum by an integral and we introduced the shorthand $\bar{n} \equiv \langle n \rangle$. This formula is valid at any temperature as long as a large number of states is involved. In the special case of zero temperature we have that $\bar{n}$ is 1 for low energy and 0 for high energy (as noted above) so we find

$$\int_0^{\epsilon_{\mathrm{F}}} g(\epsilon)\mathrm{d}\epsilon = \mathrm{N}. \tag{10}$$

The density of states typically has a simple power-law dependence on energy,

$$g(\epsilon) = D\epsilon^q \tag{11}$$

where D depends on the volume or area of the gas, and q is a constant (see eqn (8)). For a three-dimensional gas of non-relativistic particles, $q = 1/2$. By performing the integral one finds

$$\epsilon_{\mathrm{F}} = \left[(q+1)N/D\right]^{1/(q+1)} \tag{12}$$

For spin-half particles of mass m in a box of volume V, one has $D = (2m)^{3/2}V/(2\pi^2\hbar^3)$ and then

$$\epsilon_{\mathrm{F}} = \frac{\hbar^2}{2m}\left(3\pi^2 n\right)^{2/3} \qquad \text{where } n = N/V \tag{13}$$

which agrees with (4).

2 Energy, pressure, heat capacity

Once we have learned the value of ϵ_{F} (either by calculation or by deducing it from experimental observations) we can also find the total energy of the gas at zero temperaure:

$$U = \int_0^{\epsilon_{\mathrm{F}}} \epsilon g(\epsilon)\mathrm{d}\epsilon = \frac{\mathrm{D}}{\mathrm{q}+2}\epsilon_{\mathrm{F}}^{\mathrm{q}+2}, \tag{14}$$

where we used (11) for convenience. Similarly,

$$N = \int_0^{\epsilon_{\mathrm{F}}} \mathrm{d}\epsilon = \frac{\mathrm{D}}{\mathrm{q}+1}\epsilon_{\mathrm{F}}^{\mathrm{q}+1} \tag{15}$$

so

$$\frac{U}{N} = \frac{q+1}{q+2}\epsilon_{\mathrm{F}}. \tag{16}$$

Note that this way of expressing the result is convenient because we don't need to worry about the expression for D: we only need to know q. We find that at zero temperature the mean energy per particle is of the order of the Fermi energy. This is quite a lot of energy, in that it is infinitely larger than $k_{\mathrm{B}}T$, and well above the energy of the lowest-lying single-particle states.

We shall consider small but non-zero temperatures in section 2.3.

An important example of a Fermi gas at low temperature (where by low we mean $k_{\mathrm{B}}T \ll \epsilon_{\mathrm{F}}$) is the set of conduction electrons in an ordinary metal. These electrons move remarkable freely, even in the presence of the ions and the other electrons making up the metal, for largely quantum-mechanical reasons. If the ions form a regular lattice then the wave-like nature of electrons permits them to propagate long distances (of order one centimetre) without scattering off the lattice. The positive charge of the ions also shields the electrons from one another, and the Pauli Exclusion Principle further reduces their scattering, because only scattering into unoccupied states of motion is possible. For such a gas the number density is high—about 1000 times higher than an ordinary gas at standard temperature and pressure—and the particles have low mass. Consequently the Fermi energy is high, about 7 eV for copper. The Fermi temperature $\epsilon_{\mathrm{F}}/k_{\mathrm{B}}$ is then about 80000 K so at room temperature the free electrons in copper are very nearly all in the lowest available motional states. Such a gas is said to be 'highly degenerate'.

Further examples of a degenerate Fermi gas (to good approximation) occur in astrophysics, in the behaviour of white dwarfs and neutron stars, and in ultra-low-temperature experimental atomic physics.

2.1 Pressure

We can obtain the pressure of a gas (at fixed N) by using $\mathrm{dU} = \mathrm{TdS} - \mathrm{pdV}$ which gives

$$p = - \left.\frac{\partial U}{\partial V}\right|_S . \tag{17}$$

At temperatures above zero this formula is not convenient because one needs an expression for U in terms of entropy and volume in order to evaluate it. However at absolute zero temperature the entropy stays fixed (at zero) when the temperature stays fixed (at zero), so to find the pressure we just need to differentiate (16) with respect to volume. One finds

$$p = -\frac{q+1}{q+2} N \frac{\mathrm{d}\epsilon_{\mathrm{F}}}{\mathrm{d}V} . \tag{18}$$

For the 3-dimensional gas, $q = 1/2$ and $\epsilon_{\mathrm{F}} \propto V^{-2/3}$ so $\mathrm{d}\epsilon_{\mathrm{F}}/\mathrm{dV} = -(2/3)\epsilon_{\mathrm{F}}/\mathrm{V}$ and one finds

$$p = \frac{2}{5}\frac{N}{V}\epsilon_{\mathrm{F}} . \tag{19}$$

For copper at room temperature we have $N/V = 8.5 \times 10^{28}\,\mathrm{m}^{-3}$ and $\epsilon_{\mathrm{F}} \simeq 7\,\mathrm{eV}$, and one finds $p \simeq 3.8 \times 10^{10}\,\mathrm{Pa}$, which is about 380000 atmospheres! The bulk modulus of copper is approximately $1.4 \times 10^{11}\,\mathrm{Pa}$. We deduce that the electron pressure makes a non-negligible contribution to this.

The pressure in a Fermi gas at low temperature is called *degeneracy pressure*. It arises because the particles cannot all go into the ground state so they have to occupy higher states, and the energies of those states rise when the volume decreases. This pressure is very important in the interior of stars such as white dwarfs and neutron stars.

Two other ways of writing (19) are worth noting:

$$pV = N(2/5)\epsilon_{\rm F} \tag{20}$$

which reminds us of the ideal gas equation of state, except that on the right is $(2/5)\epsilon_{\rm F}$ instead of $k_{\rm B}T$. Also,

$$p = (2/5)n\,\epsilon_{\rm F} \tag{21}$$

which reminds us that pressure is proportional to energy density.

2.2 Physical intuition

When we study gasses at low temperature, we find it useful to focus in the first instance on the energy eigenstates of the quantum entities (atoms or electrons or molecules), and these states are very wave-like. But one can also think of the system in more particle-like terms. For, we don't have to assume each particle is in an energy eigenstate at any given time. Rather, each particle may be in a superposition made up of a group of states close in energy to one another. Among the possible superpositions are those in which the wavefunction, as a function of position, has the form of a *wave packet*, which is a function having a localised peak of some width Δx, and a spread of momentum values Δp, satisfying Heisenberg's uncertainty principle $\Delta x \Delta p \ge \hbar/2$. In this picture, therefore, the quantum entities are quite particle-like and we can picture them as moving about the container, colliding with the walls, and so on. This gives a good insight into the degeneracy pressure, for example.

2.3 Heat capacity

So far our discussion of the Fermi gas has been entirely concerned with the situation at $T = 0$. We shall now explore briefly what happens for $0 < T \ll T_{\rm F}$ where $T_{\rm F}$ is the Fermi temperature defined by $T_{\rm F} \equiv \epsilon_{\rm F}/k_{\rm B}$. Nothing in particular happens at the Fermi temperature, but when the temperature is of this order the Fermi-Dirac distribution function has changed substantially from its $T = 0$ shape (see Fig. 1).

The effect of a small increase in temperature above zero can be seen by comparing the curves shown in Fig. 1. Much of the distribution is unaffected, but the part of the distribution within about $k_{\rm B}T$ of the Fermi energy changes, such that the particles in this region are moved to energies about $k_{\rm B}T$ more than they had at $T = 0$. It follows that the energy of the gas increases from its value at $T = 0$ by

$$\Delta U \simeq (g(\epsilon_{\rm F})k_{\rm B}T)(k_{\rm B}T) \tag{22}$$

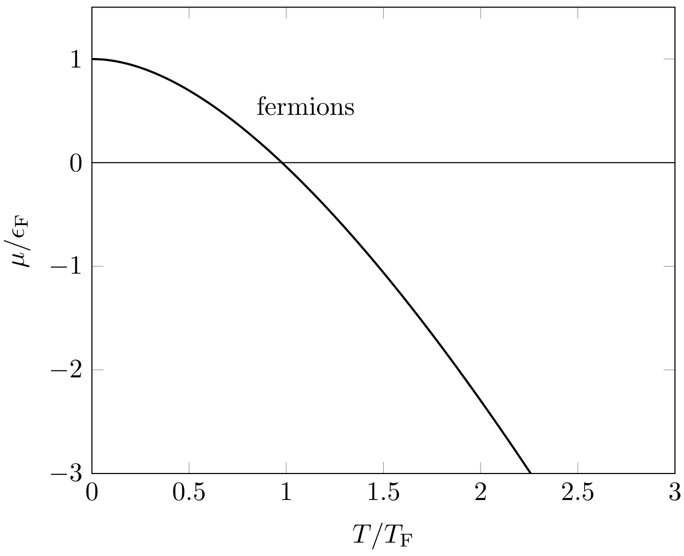

Figure 3: Behaviour of the chemical potential as a function of temperature, for fermions in 3d. $\epsilon_{\rm F}$ is the Fermi energy and $T_{\rm F} \equiv \epsilon_{\rm F}/k_{\rm B}$. $\mu = 0$ at $T \simeq 0.9887\,T_{\rm F}$. (The precise value is $4(3\sqrt{\pi}(2-\sqrt{2})\zeta(3/2))^{-2/3}T_{\rm F}$.)

where the first factor gives the number of particles affected, and the second gives their energy gain per particle. Hence we find $\Delta U \simeq g(\epsilon_{\rm F})k_{\rm B}^2T^2$ and therefore the constant-volume heat capacity is

$$C_V \simeq 2g(\epsilon_{\rm F})k_{\rm B}^2T. \tag{23}$$

Using $g(\epsilon) = D\epsilon^q$ and (15) we have

$$g(\epsilon_{\rm F}) = D\epsilon_F^q = \frac{(q+1)N}{\epsilon_{\rm F}}. \tag{24}$$

Hence for the three-dimensional gas (where $q = 1/2$) we obtain

$$C_V \simeq 3Nk_{\rm B}T/T_{\rm F}. \tag{25}$$

A more accurate calculation has to account for the fact that the chemical potential is itself a function of temperature (Figure 3 and eqn (43)),

$$\mu \simeq \epsilon_{\rm F}\left(1-(\pi^2/12)(T/T_{\rm F})^2+\cdots\right). \tag{26}$$

After performing an integral over the distribution function one finds

$$C_V \simeq (\pi^2/2)Nk_{\rm B}T/T_{\rm F}. \tag{27}$$

Notice the presence of the 'Fermi factor' $(T/T_{\rm F})$ in these expressions. The intuition is that physical effects which involve exciting particles to higher energy can only involve the particles whose energies are within about $k_\mathrm{B}T$ of the Fermi energy. The particles with energy lower than that cannot be excited by thermal effects because the states above them are full: the excitation is 'blocked'; particles with energy more than a few $k_\mathrm{B}T$ above ϵ_F also don't contribute much because there are hardly any of them.

For a metal at room temperature this heat capacity is small compared to the one associated with vibrations of the lattice (which is of order $3Nk_\mathrm{B}$). At low temperature, on the other hand, it can be the dominant contribution. At low temperature the lattice contribution is proportional to T^3 so the total heat capacity (the sum of the electron and lattice contributions) is predicted to have a temperature-dependence of the form

$$C_V = \gamma T + \alpha T^3 \tag{28}$$

where γ and α are constants specific to each metal. For this reason the experimental data is often presented as a plot of C_V/T vs T^2; one finds then a straight line whose intercept gives the electron contribution.

2.4 A more precise estimate: Sommerfeld expansion

In this section we obtain the formulae which were quoted without derivation in the previous section. This could be omitted on first reading, or for an introductory treatment.

When calculating properties such as internal energy at low temperature, the expression to be calculated often takes the form

$$I = \int_0^\infty \phi(\epsilon) f(\epsilon) \mathrm{d}\epsilon \tag{29}$$

where $f(\epsilon) = 1/(\exp(\beta(\epsilon - \mu) + 1)$ is the Fermi-Dirac distribution function. For example one would take $\phi(\epsilon) = \epsilon g(\epsilon)$ in order to calculate the internal energy. At any given temperature it is useful to write the integral I in the form

$$\int_0^\infty \phi(\epsilon) \left[f_0(\epsilon) + f(\epsilon) - f_0(\epsilon) \right] \mathrm{d}\epsilon \tag{30}$$

where $f_0(\epsilon)$ is the function

$$f_0 = \left\{ \begin{array}{ll} 1 & \epsilon < \mu \\ 0 & \epsilon \geq \mu \end{array} \right. \tag{31}$$

(c.f. Fig. 4). This f_0 is quite like the Fermi-Dirac distribution at absolute zero, but not exactly like it because $\mu \neq \epsilon_\mathrm{F}$ when the temperature is non-zero. We now introduce the notation

$$\Delta f(\epsilon) \equiv f - f_0 \; = \left\{ \begin{array}{cl} f(\epsilon) - 1 & \epsilon < \mu \\ f(\epsilon) & \epsilon \geq \mu \end{array} \right. \tag{32}$$

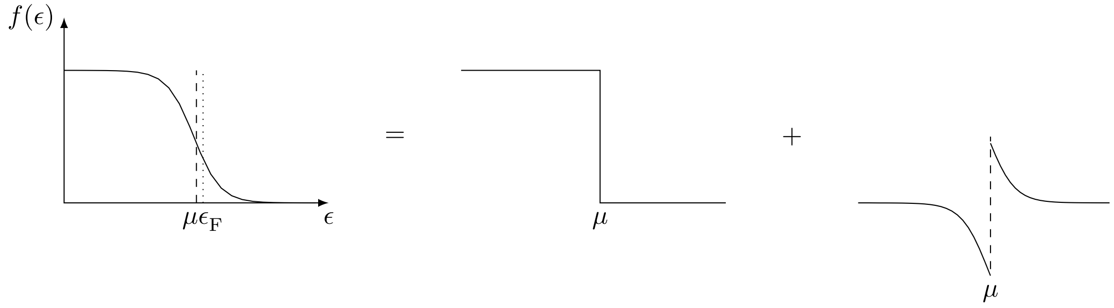

Figure 4: Showing how the distribution function can be expressed, at any given μ, as the sum of a step function and a function which is symmetric (odd) about $\epsilon = \mu$. The step is at μ, where the distribution function has the value 1/2. The decomposition is exact; μ is not equal to ϵ_F (except at absolute zero).

Hence we have

$$\begin{aligned} I &= \int_0^{\mu} \phi(\epsilon)\mathrm{d}\epsilon + \int_0^{\infty} \phi(\epsilon)\Delta\mathrm{f}(\epsilon)\mathrm{d}\epsilon && (33) \\ &= I_0 + \int_{\epsilon_F}^{\mu} \phi(\epsilon)\mathrm{d}\epsilon + \frac{1}{\beta}\int_{-\beta\mu}^{\infty} \phi\left(\frac{\mathrm{x}}{\beta}+\mu\right)\Delta\mathrm{f}\left(\frac{\mathrm{x}}{\beta}+\mu\right)\mathrm{dx} && (34) \end{aligned}$$

where

$$I_0 = \int_0^{\epsilon_F} \phi(\epsilon)\mathrm{d}\epsilon \tag{35}$$

and we introduced the change of variable $x = \beta(\epsilon - \mu)$ in the final integral.

So far no approximation has been made: eqn (34) will yield I (as defined by (29)) precisely. Now we introduce the following approximations:

1. Allow the lower limit $-\beta\mu$ on the final integral to go to $-\infty$
2. Taylor-expand the function ϕ about $\epsilon = \mu$

(these are exact at zero temperature). The second approximation is motivated by the fact that Δf is concentrated at energies close to μ (c.f. Fig. 4). The Taylor expansion of ϕ is

$$\phi(\epsilon) \simeq \phi(\mu) + (\epsilon - \mu)\phi' + \frac{(\epsilon-\mu)^2}{2}\phi'' + \ldots \tag{36}$$

where the derivatives are to be evaluated at $\epsilon = \mu$. Now observe that the function Δf is exactly an odd function about the point $\epsilon = \mu$, since

$$f(\mu + \delta) = 1 - f(\mu - \delta) \tag{37}$$

(which the reader should verify). It follows that the even terms of the Taylor expansion of ϕ will not contribute to the final integral in (34), and the linear term gives

$$\frac{\phi'}{\beta^2} 2 \int_0^\infty \frac{x}{e^2 + 1} \mathrm{dx} \;=\; \frac{\pi^2}{6} \frac{\phi'}{\beta^2}. \tag{38}$$

By substituting this into (34), and using

$$\int_{\epsilon_\mathrm{F}}^{\mu} \phi(\epsilon) \mathrm{d}\epsilon = (\mu - \epsilon_\mathrm{F}) \phi(\mu) + \mathcal{O}\left((\mu - \epsilon_\mathrm{F})^2\right) \tag{39}$$

we have

Sommerfeld expansion

$$I \simeq I_0 + (\mu - \epsilon_\mathrm{F})\,\phi + \frac{\pi^2}{6} (k_\mathrm{B} T)^2 \frac{\mathrm{d}\phi}{\mathrm{d}\epsilon} \tag{40}$$

where I_0 is the value of the relevant quantity at absolute zero, and ϕ and ϕ' should be evaluated at μ.

The above method will adopt $\phi = g$ when when applied to the particle number, and $\phi = \epsilon g$ when applied to the internal energy. One thus obtains:

$$N \;=\; N_0 + (\mu - \epsilon_\mathrm{F})\, g(\mu) + \frac{\pi^2}{6} (k_\mathrm{B} T)^2 \frac{\mathrm{d}g}{\mathrm{d}\epsilon} \tag{41}$$

$$U \;=\; U_0 + (\mu - \epsilon_\mathrm{F})\, \mu g(\mu) + \frac{\pi^2}{6} (k_\mathrm{B} T)^2 \frac{\mathrm{d}}{\mathrm{d}\epsilon} (\epsilon g). \tag{42}$$

Using $N = N_0$, the first result yields

$$\mu = \epsilon_\mathrm{F} - \frac{\pi^2}{6} (k_\mathrm{B} T)^2 \frac{1}{g} \frac{\mathrm{d}g}{\mathrm{d}\epsilon}, \tag{43}$$

which, for the case of a gas in a box in three dimensions, gives (26). By substituting this into (42) we then obtain

$$U = U_0 + \frac{\pi^2}{6} (k_\mathrm{B} T)^2 g(\mu) \tag{44}$$

which, applied to a gas in a box in three dimensions, yields (27).

By writing the expressions in terms of g, we have ensured that all the equations in this section apply to a Fermi gas in a box of any number of dimensions, and also of any kind of potential energy as a function of position. All one needs is the density of states. By using integrals we have adopted a continuous approximation to the density of states. This is a good approximation when the Fermi energy is large compared to the energy gap, which is the case when N is large.

3 Degeneracy pressure in stars

The electron density in the core of the sun is $6.4 \times 10^{31}\,\mathrm{m}^{-3}$ and the temperature is 16 million kelvin. Using (13) we find the Fermi energy is 583 eV. Therefore the temperature is about 2.3 times $\epsilon_\mathrm{F}/k_\mathrm{B}$. This shows that the plasma in the solar core can be treated as a classical gas to first approximation, but it would only require a modest drop in temperature (at given density) to cause the electrons to enter the low-temperature regime where degeneracy pressure is important.

Stars such as the Sun eventually exhaust their nuclear fuel and as a result the temperature falls, and the pressure with it, so the star begins to collapse to a smaller size than it had when hot. The pressure will then build up again owing to the higher density as the star shrinks. The collapse will be halted if the pressure rises sufficiently to balance the gravitational forces. Let us estimate the condition for this to occur.

With a view to getting the right order of magnitude, we treat the star as a sphere of uniform density. Then the total gravitational binding energy is

$$E_\mathrm{grav} = -\frac{3}{5}\frac{GM^2}{R} \tag{45}$$

where M, R are the mass and radius of the star. For $T \ll T_\mathrm{F}$ the total kinetic energy of the electrons is given by (16):

$$U = \frac{3}{5}\epsilon_\mathrm{F} N = \frac{3}{5}\frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{(4/3)\pi R^3}\right)^{2/3} N. \tag{46}$$

Hence the total energy has the form

$$E = E_\mathrm{grav} + U = -\frac{A}{R} + \frac{B}{R^2} \tag{47}$$

where all the constants have been bundled into A and B. For a given mass, the minimum total energy is then obtained when

$$R = \frac{2B}{A} = \frac{\hbar^2}{8mm_\mathrm{p}^2 G}\,(9\pi)^{2/3}\,m_\mathrm{p}^{1/3} M^{-1/3} \tag{48}$$

where m_p is the mass of a proton and we took $N = M/2m_\mathrm{p}$ which is a good estimate if the star is composed mostly of elements such as helium, carbon and oxygen whose atomic weight is twice their atomic number (the number of electrons being equal to the number of protons in a neutral plasma). Notice that the radius falls as the mass increases, and consequently the density is proportional to M^2. For one solar mass we find $R \simeq 7000\,\mathrm{km}$ (similar to that of the Earth) and the density is one tonne per cm^3. (We have become accustomed to such numbers but when they were first inferred from astronomical evidence it was doubted that such a state of matter could exist.)

A star which is supported in this way by electron degeneracy pressure, with little or no nuclear fusion, is called a *white dwarf*. Such a configuration is the end-point of the evolution of many stars; indeed 97% of those in the Milky Way galaxy are thought to end this way.

A white dwarf of one solar mass has $\epsilon_{\rm F} \simeq 0.2\,$MeV, which is about two fifths of the electron rest energy, which means the electrons have speeds equal to a non-negligible fraction of the speed of light. To describe a somewhat more dense star we shall require, therefore, a relativistic treatment. The density of states in k-space remains the same, so we have $g_k = k^2 V/\pi^2$ and $k_{\rm F} = (3\pi^2 n)^{1/3}$ as before, but the relation between energy and momentum is now $E^2 = m^2c^4 + p^2c^2 = m^2c^4 + (\hbar kc)^2$, leading to

$$\begin{aligned} g(E) &= \frac{VE}{\pi^2(\hbar c)^3}\sqrt{E^2 - m^2c^4} \;\simeq\; \frac{VE^2}{\pi^2(\hbar c)^3} && (49)\\ E_{\rm F} &= mc^2\left(1 + (\hbar/mc)(3\pi^2 n)^{2/3}\right)^{1/2} \;\simeq\; \hbar c(3\pi^2 n)^{1/3}, && (50)\end{aligned}$$

where in each case the approximate version gives the ultra-relativistic limit. In that limit the Fermi energy now scales as $1/R$ instead of $1/R^2$ and consequently the total kinetic energy scales with R in the same way as the gravitational potential energy. In place of (47) we have

$$E = \frac{B - A}{R} + O(R) \tag{51}$$

where $A = (3/5)GM^2$ as before, and now

$$B = \frac{3}{4}E_{\rm F}N = \frac{3}{4}\hbar c\left(\frac{9\pi}{4}\right)^{1/3}\left(\frac{M}{2m_{\rm p}}\right)^{4/3}. \tag{52}$$

When $B > A$ the star can find a minimum-energy state at some R where the R and $1/R$ terms balance one another. When $B < A$, on the other hand, there is no equilibrium point and the star shrinks without limit—or until some other process not accounted-for in the above calculation takes hold. The stability condition $B > A$ places an upper limit on the mass. For a more massive star, electron degeneracy pressure is not sufficient to prevent gravitational collapse. In our rough model this limit is given by

$$M = \frac{3\sqrt{125\pi}}{64}\left(\frac{\hbar c}{G}\right)^{3/2}\frac{1}{m_{\rm p}^2} \;\simeq\; M_\odot. \tag{53}$$

A calculation bringing in a model of the density and pressure distribution in hydrostatic equilibrium leads to $1.4M_\odot$. This is known as the *Chandrasekhar limit.*

Stars of mass exceeding the Chandresakhar limit may undergo a process leading to a supernova explosion, or they may shrink to a neutron star or to a black hole. The neutron star is a star supported largely by degeneracy pressure once again, but now from neutrons rather than electrons. A calculation similar to the above is little changed, except that now $N \simeq M/m_{\rm p}$ so a factor $2^{-4/3}$ is omitted from B and consequently the limiting mass increases by a factor 4. A more thorough calculation gives a value called the *Tolman–Oppenheimer–Volkoff limit* which is around 2 to 3 $M_\odot$.

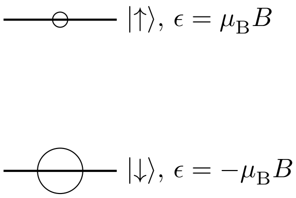

Figure 5: The two spin states available to a spin-half particle, and their energies in the case of an electron in a magnetic field. At lower energy the magnetic dipole aligns with the field and therefore the spin state is anti-aligned with the field. The circles represent illustrative occupation probabilities in thermal equilibrium for an isolated electron.

4 Free electron paramagnetism

When a magnetic field is applied to a material which is non-magnetic in the absence of an applied field, the material will typically acquire a magnetic moment. There are two main mechanisms whereby this occurs: the material may be composed of a large number of small magnetic dipoles (either single particles or small domains) which will tend to line up with one another under the influence of the applied field. This results in a magnetization in the direction of the applied field and is called *paramagnetism*. But there is always a competing process, associated with the response of moving charges to an applied field, and if this results in magnetization in the opposite direction to the applied field then it is called *diamagnetism*.

In the case of free electrons, both processes are relevant. We shall calculate the paramagnetic contribution and then quote the diamagnetic contribution.

Each electron carries an intrinsic magnetic dipole moment of size and direction equal to

$$\mathbf{m} = -g_s \mu_{\rm B} \hat{\mathbf{s}} \tag{54}$$

where $g_s \simeq 2.002$ and $\hat{\mathbf{s}}$ is the electron spin operator in dimensionless units (it is the intrinsic angular momentum divided by $\hbar$) and $\mu_{\rm B}$ is the standard atomic unit of magnetic dipole moment, called the *Bohr magneton*:

$$\mu_{\rm B} = \frac{e\hbar}{2m}. \tag{55}$$

This is not to be confused with chemical energy! Under the influence of an applied magnetic field, the hamiltonian describing the interaction between the dipole and the field is

$$H = -\mathbf{m} \cdot \mathbf{B} \tag{56}$$

and since there are two spin states (called spin-up and spin-down) having $s_z = \pm 1/2$, respectively, there are two eigenstates of energy, with energies $\mp \mu_{\rm B} B$ (where we took $g_s = 2$ for simplicity); see Fig. 5.

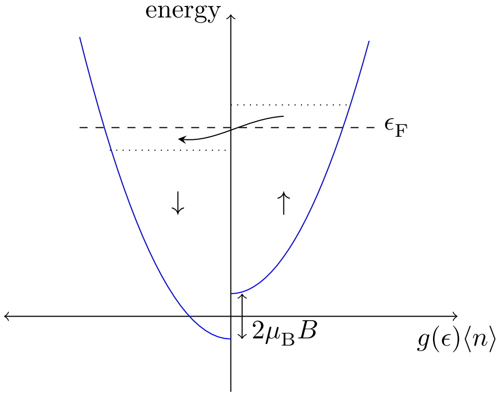

Figure 6: In the presence of a magnetic field the energy levels of free electrons are shifted, those of spin up to higher energy, those of spin down to lower. At zero temperature the states with energy below the Fermi energy are occupied. Compared to the situation at zero field, the states indicated by the curved arrow switch orientation.

If we were dealing with distinguishable particles we could now consider each electron separately. By invoking the Boltzmann distribution one would find the probabilities to be spin-up and spin-down are in the ratio $p_+/p_- = \exp(-2\beta\mu_{\rm B}B)$. Then, after using also $p_+ + p_- = 1$ one finds the mean magnetic dipole moment per electron is

$$\langle m\rangle = \mu_{\rm B}\left((+1)p_- + (-1)p_+\right) = \mu_{\rm B}\tanh(\beta\mu_{\rm B}B) \tag{57}$$

and therefore the total dipole moment is

$$m_{\rm tot} = N\mu_{\rm B}\tanh(\beta\mu_{\rm B}B). \tag{58}$$

For temperatures large compared to $\mu_{\rm B}B$ this predicts that the electrons all line up with the field, giving a total magnetic dipole moment of $N\mu_{\rm B}$ which is large for N of the order of Avogadro's number: the field produced by such a total dipole is of order one tesla.

Now let's consider the highly degenerate case. We will first consider the limit as $T \to 0$, and then consider finite temperature. A gas of free electrons (such as in a metal, where we neglect electron interactions) at low temperature has the Fermi-Dirac distribution. In the presence of a magnetic field the energies available to the electrons are given by the sum of their kinetic and magnetic energy. This leads to two distributions, one for the spin-up states, one for the spin-down:

$$\begin{aligned}\langle n_\uparrow\rangle &= \frac{1}{e^{\beta(\epsilon+\mu_{\rm B}B-\mu)}+1}\\ \langle n_\downarrow\rangle &= \frac{1}{e^{\beta(\epsilon-\mu_{\rm B}B-\mu)}+1}.\end{aligned} \tag{59}$$

Here we have adopted a notation where ϵ signifies just the kinetic energy, so at each ϵ there are two quantum states, one of energy $\epsilon + \mu_{\rm B}B$ and the other of energy $\epsilon - \mu_{\rm B}B$. Note that there is just one chemical potential μ describing the whole system, and one $\beta = 1/k_{\mathrm{B}}T$. A good insight into the situation is to be had by plotting the above two functions on a single diagram, having first multiplied by the density of motional states $g(\epsilon) = D\epsilon^{1/2}$. This is shown in Fig. 6. This diagram is somewhat like Fig. 1 but differs in two respects: we have chosen to plot $g(\epsilon)\langle n\rangle$, not $\langle n\rangle$ alone, and we have swapped the horizontal and vertical axes. Now energy is shown in the vertical direction on the diagram, and a nice feature is that now the area bounded by the curves on the diagram is proportional to the number of quantum states up to the given energy. The Fermi energy is that energy below which there are N states altogether. At zero temperature the occupied states are those below the Fermi energy. In the presence of a magnetic field the distributions shift in energy, as shown, with the result that some electrons switch spin-state from up to down in order to occupy the lower-energy states. For low B field the number switching is $(1/2)g(\epsilon_{\mathrm{F}})\mu_{\mathrm{B}}B$, (the half is needed because we have divided the distribution into two parts), resulting in a total dipole moment for the whole set of electrons

$$m_{\mathrm{tot}} = g(\epsilon_{\mathrm{F}})\mu_{\mathrm{B}}^2 B. \tag{60}$$

We thus find that the magnetization is only owing to those electrons within about μB of the Fermi energy, which is a small fraction of the total. The magnetization is very much smaller than that predicted by (58) and is almost independent of temperature, since (60) remains a good approximation for $T \ll \epsilon_F/k_{\mathrm{B}}$.

So far we discussed paramagnetic effects. The quantum theory of the coupling of a moving charge to a magnetic field leads to a further contribution to the energy. That effect is diamagnetic and of a size one third of the paramagnetic effect discussed above, so the overall magnetization is two-thirds of that predicted by (60).