

Atomic Physics (OUP 2005)

C.J. Foot, Oxford, 9 February 2005

Chapter 1

(1.1) 0.18 nm

(1.2) Balmer series in hydrogen and transitions to $n = 4$ shell in He^+ . Energy $\propto Z^2/n^2$. Lines of similar wavelength show isotope shift: wavelength ratio H/He is 1.0004 (equal to ratio of reduced masses given by eqn 1.13).

(1.3) From eqns 1.17 and 1.18:

$$\Delta E = \frac{\alpha^2}{n^2} E.$$

The $n = 2$ shell has $E = hcR_\infty = 3.4$ eV, hence $\Delta E = 4.5 \times 10^{-5}$ eV which requires resolution greater than $E/\Delta E = 75\,000 = 4/\alpha^2$.

(1.5) K-absorption edges: Mn 6.54 keV; Fe 7.11 keV. Good contrast at 6.8 keV. [data from <http://www-structure.llnl.gov/xray/elements.html>]

(1.6) www.physics.ox.ac.uk/history.asp?page=Exhibit10

(1.7) M-shell absorption is about 3.8 keV (see Ex 1.5) which implies $\sigma_M \simeq 32$, but any reasonable guess is acceptable. Estimate relativistic effects to be a few %, or higher.

(1.9) 7.9×10^4 K

(1.10) $\mu_B B = 14$ GHz for $B = 1$ T. Light of wavelength $\lambda = 600$ nm has $f = 5 \times 10^{14}$ Hz, hence $\Delta f/f = 3 \times 10^{-5}$. Earth's field is about 5×10^{-5} T (in the UK).

Chapter 2

(2.2) Apply ladder operator.

(2.5) Isotope shift: 124 GHz (see Ex 1.1); for fine structure splitting and Lamb shift see Section 2.3.4. An etalon of length 1 cm and finesse 100 has transmission peaks whose FWHM = 0.15 GHz (assuming air between mirrors). Thus it is easy to find an instrument with

high enough resolution but in practice measurements are limited by Doppler broadening (eqn 6.38); (a) isotope shift is fully resolved, (b) fine structure is just resolved, and (c) the Lamb shift cannot be resolved.

(2.6) (a) vacuum ultraviolet, 0.45 ns, and (b) near infra-red, 450 ns (using eqn 1.24).

(2.10) (c) $\mathcal{I}_{\text{rad}}\mathcal{I}_{\text{ang}} = 0.28 a_0$ (to be checked).

(d) Bulge in xy -plane rotating around the z -axis.

(e) π -transitions related to linear dipole oscillating along z -axis. σ -transitions related to circular motion in xy -plane.

(2.13) Excitation to $n = 5, l = 4$ configuration and subsequent decay to $n = 4, l = 3$; $n = 3, l = 2$; $n = 2, l = 1$. (Lyman- α not detected.)

Chapter 3

(3.1) (b) Binding energy of an electron, $4 \times 13.6 = 54.4$ eV. (c) For given separation the repulsive energy equals the binding energy so estimated I.E. would be zero; this is not a small perturbation and the repulsive energy needs to be calculated more carefully as described in the text. (Clearly the mean separation is greater than r .) (d) Ignoring repulsion, binding energy is $14^2 \times 13.6 = 2667$ eV (which is 11% higher than expt.) Including repulsion (proportional to Z) gives 2286 eV (which is a 5% lower than expt.) The repulsion is less important for high- Z atoms (relative to the attraction to the nucleus).

Chapter 4

(4.1) $1s^2 2s^2 2p^6 3s^2 3p^6 3d^{10} 4s^2 4p^6 4d^{10} 4f^{14} 5s^2 5p^6 5d^{10} 6s^2 6p^6 7s$

(4.2) List in ascending order: 38541, 39299, 39795, 40137, 40383, 40566, 40706, 40814. Plot graph of differences in wavenumber between each pair against wavenumber (highest value for each pair); extrapolate to find where difference goes to zero ($\sim 41\,250$ cm^{-1}). I.E.(Na) given in Table 4.1, and quantum defect etc. in Table 4.2.

(4.3) See Table 4.2. The 3s configuration has $n^* = \sqrt{13.606/5.14} = 1.63$. Between 3s and 6s the quantum defect decreases by 1.5%. The 8s configuration has binding energy 0.31 eV (assuming same quantum defect as for 6s), c.f. 0.21 eV for the $n = 8$ shell of hydrogen.

(4.4) Quantum defect for 5s configuration is 3.19 (calculate, or look at Table 4.2); use this value to estimate energy of 7s configuration. Taking difference in energy between 5s and 7s, dividing by 2 gives an energy equivalent to a wavelength of 767 nm.

(4.5) (a) 625 nm from data given, c.f. 656 nm for Balmer- α . Outer electron in helium has similar energy to that in hydrogen. [Actually the line in helium has a wavelength of 668 nm; estimate not accurate because of averaging singlet and triplets.] (b) For the lowest configuration $n^* = \sqrt{R_\infty/35250} = 1.76$ and $\delta = n - n^* = 0.24$. The given configurations have $\delta = 0.24, 0.028, 0.23, 0.029, 0.003$, i.e. $\delta_s > \delta_p > \delta_d$. (c) 6859 cm^{-1} . (d) Binding energy of 4f configuration in Li^+ is $hcR_\infty/4 = 3.4 \text{ eV}$. Answer given.

(4.6) See previous Exercise and eqn 4.13.

(4.7) Note error in Exercise: fine structure splitting in neutral sodium (0.002 eV) and hydrogen ($1.3 \times 10^{-5} \text{ eV}$). Fine-structure splitting in Na^{+10} is $Z^4 = 14641$ times that of the same configuration in H, namely 0.2 eV. Value for neutral atom is approximately the geometric mean of the other two, i.e. fine structure of neutral atoms scales as Z^2 .

(4.8) (b) Ratio 1 : 20 : 14.

(4.9) Note obvious error in Exercise: should be sum from $m_l = -l$ to l .

Chapter 5

(5.3) Interval rule implies levels $J = 0, 1, 2$ belonging to a ^3P term. Allowed transition to a ^3S term that has no fine structure (its only level has $J = 1$).

First three wavenumbers listed obey an interval rule that indicates levels $J = 1, 2, 3$ (^3D term).

Other levels must belong to ^3P , ^3D or ^3F (from selection rules). Sketching the energy levels and allowed transitions shows that the same ^3P term as in the first part fits the data (with levels $J = 0, 1, 2$ and intervals between them of 52 and 106 cm^{-1} respectively). The fine structures of the terms in this example obey interval rule to within a few %. Possible further experiments: observe the anomalous Zeeman effect and count the number of components into which the line is split (c.f. Fig. 5.13) to deduce J_{lower} and J_{upper} ; one could also measure g_J (see Exercise 5.8).

(5.4) Lowest term $^1\text{S}_0$. First three excited levels are an obvious triplet whose spacing obey the interval rule ^3P with levels $J = 0, 1, 2$. Next come the terms ^1P , ^3S and ^1S , none of which have fine structure (c.f. magnesium in Fig. 5.9). [N.B. There is a numerical coincidence which leads to the ratio of another pair of intervals being almost exactly 2.]

(5.5) The LS -coupling scheme gives an accurate description of the Mg atom, but is less good for the Fe ion (ratio of intervals equals about 2.5, rather than 2 as expected from interval rule), hence $\Delta S \neq 0$ transitions observed in spectrum of the ion but such intercombination lines are NOT observable in Mg.

(5.6) Hund's rules and magnetism are described in Blundell (2001): his Table 3.1 gives the magnetic ground states for 3d ions.

(5.7) The electrons in the low-lying configuration have a residual electrostatic energy much greater than the spin-orbit interaction (of the 3p electron), hence the LS -coupling scheme is a good approximation. Electrons in the higher configuration are 'further apart' (smaller overlap of their wavefunctions leading to a smaller exchange integral) and the residual electrostatic interaction is smaller than the spin-orbit interaction of the 3p-electron, thus the jj -coupling scheme is appropriate and the levels are in two doublets. The $J = 1$ levels are mixed.

(5.8) The change over from the LS - to jj -coupling scheme occurs because the spin-orbit interac-

tion increases relative to residual electrostatic interaction (whereas in the previous Exercise this arose because $E_{r.e.}$ decreased). For a $J = 1$ level of a pure 1P term (i.e. a term for which the LS -coupling scheme is very well obeyed) has $g_J = 1$ (spin equals zero). This value is close to the given g -factor hence assume that it is this level and there is some mixing with the wavefunction of 3P_1 .

- (5.9) (a) No (b) No (c) Yes (d) No (e) No.

The $4d^95s5p$ configuration has a hole in the d -shell (that on its own would give a ${}^2D_{5/2}$ level); coupling with the angular momentum of the $5s$ and $5p$ electrons gives rise to many levels including ${}^2P_{3/2}$. (Other configurations might be involved but this is the one most likely to have similar energy.)

- (5.10) N.B. No central component since $M_J = 0$ to $M_J = 0$ does not occur when $\Delta J = 0$. Six components whose relative separations (on a diagram similar to that in Exercise 5.12) would be in the ratios $\frac{1}{2} : 1 : 1 : 1 : \frac{1}{2}$, where 1 corresponds to the given frequency unit.

- (5.12) A 3P term has $J = 0, 1, 2$. Consideration of all possibilities: $J = 0 \leftrightarrow J' = 1$; $J = 1 \leftrightarrow J' = 1$; $J = 1 \leftrightarrow J' = 2$; $J = 2 \leftrightarrow J' = 2$; $J = 2 \leftrightarrow J' = 3$ shows that only $J = 1 \leftrightarrow J' = 2$ gives 9 components. A diagram similar to Fig. 5.13, shows that the Zeeman shifts of the components are (in units of $\mu_B B/h$): $(g_J - g_{J'})$, g_J , $g_{J'}$ and $2g_{J'} - g_J$ but not necessarily in that order. (We assume that the pattern is symmetric and that there is a central component with no shift.) These shifts must be in the ratio $2 : 5 : 7 : 9$. Now consider possible terms:

$$\begin{aligned} {}^3P_1 &\leftrightarrow {}^3P_2 \text{ or } {}^3D_2 \\ {}^3P_2 &\leftrightarrow {}^3S_1, {}^3P_1 \text{ or } {}^3D_1 \end{aligned}$$

Note that ${}^3P_2 \leftrightarrow {}^3P_1$ appears twice so there are only four possibilities. The g -factors are:

term	g_J
3S_1	2
3P_1	3/2
3D_1	1/2
3P_2	3/2
3D_2	7/6

Guess that 3D_2 with $g_J = 7/6$ is involved since a 7 appears in the ratios and indeed ${}^3P_1 \leftrightarrow {}^3D_2$ fits data, otherwise check all possibilities. [Straightforward but rather long if one does not stay on track.]

- (5.13) (a) $g_J = 2, 2/3, 4/3$. (b) and (c) see books: Woodgate, Rae, Cohen-Tannoudji *et al*, etc. (d) Interval of $1700 \text{ m}^{-1} = 510 \text{ GHz}$ hence $B = 510/14 = 36 \text{ T}$ is the flux density of the orbital field. [$\mu_B \equiv 14 \text{ GHz T}^{-1}$]

Chapter 6

- (6.1) 17 T, 2 T, 0.2 T.

- (6.2) $X = 91.9 \text{ MHz}$.

- (6.3) Hydrogen has a larger ground state h.f.s. than lithium because of the relatively large magnetic moment of proton and the high strength of the magnetic field at the nucleus produced by a 1s-electron (Exercise 6.1).

- (6.4) Splitting proportional to AF (interval rule), where $A \propto g_I = \mu_I/I$. The hyperfine levels in hydrogen have $F = 1$ and 2; in deuterium $F = 1/2$ and $3/2$. Thus

$$\frac{\Delta f(\text{H})}{\Delta f(\text{D})} = \frac{A_{\text{H}}}{A_{\text{D}} \times 3/2} = \frac{\mu_I/I}{\mu_{I'}/I' \times 3/2} = 4.3$$

The helium ion has the same nuclear spin as hydrogen and therefore the same angular momenta. The strength of the magnetic field at the nucleus is proportional to the square of the electron's wavefunction at $r = 0$; since $|\psi(0)|^2 \propto Z^3$ and $Z^3 = 8$ for helium, we find

$$\frac{\Delta f(\text{H})}{\Delta f(\text{He}^+)} = \frac{A_{\text{H}}}{A_{\text{He}^+}} = \frac{\mu_I}{8\mu_{I'}} = -0.16$$

- (6.5) Same values of F as in Example 6.2, hence both isotopes have same I . Since both isotopes have the same hyperfine levels we can scale the value (as in Exercise 6.2) to find 9 MHz.

- (6.6) $I = 2$.

- (6.7) The two strong components arise from the abundant isotope; this is confirmed by checking that the ratio of the total intensities: $(70 +$

42)/(5 + 3) = 14, is the same as the ratio of abundances. The ground configuration (4s) has a large hyperfine structure. The sum of the intensities to (or from) a given level is proportional to its statistical weight (2F + 1)—the same sum rule as in fine structure (Section 4.6.1). Both isotopes have intensity ratio of 5/3 and hence the levels are F = 1 and F = 2. Since J = 1/2 we deduce that I = 3/2 for both isotopes. Ratio of their nuclear magnetic moments (³⁹K to ⁴¹K): 1.6/0.9 = 1.8.

(6.8) (e) 0.2 T

(6.9) Note correction.

(6.10) About 1 ppm (1 part per million).

(6.13) Size of orbit inversely proportional to both mass and nuclear charge Z (eqn 1.9):
 $a_0/(11 \times 207) = 23 \text{ fm}$ (c.f. nuclear radius 3.4 fm). Energy proportional to Z^2 and mass:
 $hcR_\infty \times 11^2 \times 207 = 340 \text{ keV}$. Volume shift equals 4% of transition energy.

Chapter 7

(7.3) (a) See Ex. (13.5) with $|1\rangle \rightarrow |0\rangle$ and $|2\rangle \rightarrow |1\rangle$.
 (c) $\cos(\phi + \pi) = -\cos(\phi)$. (f) When $\phi = 0$ the probabilities of being in $|1\rangle$ or $|2\rangle$ are unchanged by the pulse sequence. For a system that starts in $|2\rangle$ the final probabilities are the same as given in part (e); if initially the the state is $|1\rangle$ then these probabilities are swapped.

(7.6) (b) and (c) Find the spontaneous decay rate for level i by summing A_{ij} over all allowed transitions, e.g. for the 3p sum over 1s–3p and 2s–3p.

	$\sum A_{ij}$ s ⁻¹	$1/\sum A_{ij}$ ns
3s	6.3×10^6	160
3p	1.9×10^8	5.4
3d	6.5×10^7	16
2p	6.3×10^8	1.6

For 2p, A_{21} equals the reciprocal of its lifetime. Comparison of the 1s–2p and 1s–3p transitions shows that the former has a higher A_{ij} despite having a lower frequency; this arises because of greater overlap of the 2p and 1s wavefunctions

as compared to 3p and 1s. [A simple sketch would show this clearly.]

(d)

	$A_{ij}/10^6$ s ⁻¹	g_2/g_1	$\ddagger\ddagger$	$\omega/10^{15}$ rad s ⁻¹	D a_0
2p–3s	6.3	1/3	5/36	2.9	0.54
1s–3p	170	3	8/9	18.4	0.52
2s–3p	22	3	5/36	2.9	3.0
2p–3d	65	5/3	5/36	2.9	3.9
1s–2p	630	3	3/4	15.5	1.3

Column $\ddagger\ddagger$ contains $\frac{1}{n_j^2} - \frac{1}{n_i^2}$.

$\omega = 2\pi cR_\infty(\frac{1}{n_j^2} - \frac{1}{n_i^2})$.

(e)

	$I_{\text{sat}}/\text{W m}^{-2}$
2p–3s	4.6
1s–3p	37 000
2s–3p	140
2p–3d	49
1s–2p	72 000

Chapter 8

(8.1) 2.4 GHz, 0.4 GHz.

(8.2) The Doppler width is 2.3 GHz and the two lines have a frequency difference of 10 GHz (fine structure). To be resolved the Zeeman splitting must be greater than the linewidth, but this would make it comparable with the fine structure and therefore observation of a true weak-field effect is not really possible.

(8.3) Linewidth 28 MHz.
 Collimation angle 0.014 rad.

(8.4) Hyperfine levels are $F = 3, 4$ and $F' = 2, 3, 4, 5$. The selection rule $\Delta F = 0, \pm 1$ leads to six allowed electric dipole transitions. Analysis of the frequency differences shows that B, D, E, b, d and e are cross-over resonances (and also $f_f - f_c = f_C - f_A = 201.5 \text{ MHz}$).

(a) Interval rule:

$$\begin{aligned} \Delta E_{5,4}/5 &= 251.4/5 = 50.3 \\ \Delta E_{4,3}/4 &= 201.5/4 = 50.4 \\ \Delta E_{3,2}/3 &= 151.5/3 = 50.5 \end{aligned}$$

hence $A_{6P_{3/2}} = 50.4 \text{ MHz}$.

The interval between the hyperfine levels of the ground configuration is

$$\begin{aligned} 4A_{6S_{1/2}} &= f_c - f_A \quad \text{or} \quad f_f - f_C \\ &= (f_c - f_a) + (f_f - f_A) + (f_a - f_F) \\ &= 151.5 + 452.9 + f_a - f_F \end{aligned}$$

where $f_a - f_F = 8588.2$ MHz is given on the spectrum, so that $A_{6S_{1/2}} = 2298.2$ MHz.

(b) Estimate the temperature from the Doppler width of the absorption, i.e. ignore the Doppler-free peaks and take the FWHM of the dip to be $\sim f_F - f_A \simeq 450$ MHz.

$$\begin{aligned} \Delta f_D &= 2\sqrt{\ln 2} \frac{u}{\lambda} = \frac{2\sqrt{\ln 2}}{\lambda} \sqrt{\frac{2k_B RT}{M}} \\ \Rightarrow T &= \frac{M (\Delta f_D \lambda)^2}{k_B 8 \ln 2} \simeq 400 \text{ K} \end{aligned}$$

Anything from 300-500 K is an acceptable answer. The underlying Doppler absorption profile is broadened because of hyperfine splitting of the lines and in fact these data were taken at room temperature.

(8.5) Splitting proportional to $1/n^3$ (eqn 6.10).

(8.6) (a) 0.5 MHz. (b) Difficult to guess cross-section for collisions between an atom in the ground state and an atom in the excited state that causes de-excitation of the excited atom (inelastic collision)—assuming pressure broadening of 30 GHz/bar, as in Example 8.3, would imply a contribution of 9 MHz to the linewidth at the transition frequency (and half this value at the frequency of the radiation), but see below. (c) 2 MHz. (d) Zero contribution to first order. Second order Doppler broadening $\ll 1$ MHz (c.f. eqn 8.23).

Measured width 17 MHz. Pressure shift is the dominant contribution and so it must be about 15 MHz (three times larger than in hydrogen which is not unreasonable).

(8.7) 0.7 mbar. $\sigma = 1 \times 10^{-18} \text{ m}^2$. $\sqrt{\sigma/\pi} = 6 \times 10^{-10} \text{ m}$ (c.f. the radius of the Bohr orbit for $n = 2$ is $2 \times 10^{-10} \text{ m}$); metastable hydrogen is delicate because the $2s \ ^2S_{1/2}$ and $2p \ ^2P_{1/2}$ levels lie close together in energy so that it takes only a weak perturbation to mix them.

Chapter 9

(9.1) Solar irradiance 1.4 kW m^{-2} (some of which is absorbed, or reflected, before reaching ground level but ignore this). $F_{\text{rad}} = 3.3 \times 10^{-7} \text{ N}$.

(9.2) $q_{\text{photon}} = \hbar\omega/c = \hbar k$.

(9.3) Consider conservation of energy for the emission of a photon of frequency $\tilde{\omega}$ in rest frame of the atom:

$$Mc^2 + \hbar\omega_0 = \hbar\tilde{\omega} + \tilde{\gamma}Mc^2$$

Here $\tilde{\gamma} = (1 - v^2/c^2)^{-1/2} \simeq 1 + \frac{1}{2}(v_{\text{rec}}^2/c^2)$, where v_{rec} is the recoil velocity (related to the recoil energy by $E_{\text{rec}} = \frac{1}{2}Mv_{\text{rec}}^2$). Hence

$$\begin{aligned} \hbar(\tilde{\omega} - \omega_0) &= (1 - \tilde{\gamma})Mc^2 \\ \tilde{\omega} &\simeq \omega_0 - E_{\text{rec}}/\hbar \end{aligned}$$

For an atom moving with velocity \mathbf{v}' before the emission, the frequency in the laboratory frame is related to $\tilde{\omega}$ (the frequency in the rest frame of the atom) by a Lorentz transformation; we use the transformation from the lab. frame to the rest frame so that the angle θ is between \mathbf{v}' and the wavevector \mathbf{k}_{em} (which gives the direction of photon in the laboratory):

$$\tilde{\omega} = \gamma' \omega' (1 - \frac{v'}{c} \cos \theta)$$

Since $\omega' = \omega_{\text{em}}$ and $|\mathbf{k}_{\text{em}}| = \omega'/c$

$$\tilde{\omega} = \gamma' (\omega_{\text{em}} - k_{\text{em}} v' \cos \theta)$$

Hence

$$\begin{aligned} \omega_{\text{em}} &= \frac{\tilde{\omega}}{\gamma'} + \mathbf{k}_{\text{em}} \cdot \mathbf{v}' \\ &= \left(\omega_0 - \frac{E_{\text{rec}}}{\hbar} \right) \left(1 - \frac{(v')^2}{c^2} \right)^{1/2} + \mathbf{k}_{\text{em}} \cdot \mathbf{v}' \\ &\simeq \omega_0 - \frac{E_{\text{rec}}}{\hbar} - \frac{1}{2} \omega_0 \left(\frac{v'}{c} \right)^2 + \mathbf{k}_{\text{em}} \cdot \mathbf{v}' \end{aligned}$$

One can check that the cross-terms can be neglected by estimating their magnitudes for a sodium atom with $v' \simeq 100 \text{ m s}^{-1}$:

$$\frac{E_{\text{rec}}}{\hbar\omega_0} = \frac{\hbar k^2}{2M\omega_0} = \frac{v_{\text{rec}}}{2c} \simeq 5 \times 10^{-12}$$

is the frequency shift caused by recoil (which is crucial to the argument).

$$\frac{1}{2} \left(\frac{v'}{c} \right)^2 \simeq 6 \times 10^{-14}$$

is the second-order Doppler shift equivalent to time dilation in special relativity, which is not important because it causes the same shift of the resonance frequency (towards lower frequency) in both absorption and emission, and

$$\frac{k_{\text{em}} v'}{\omega_0} = \frac{v'}{c} \simeq 3 \times 10^{-7}$$

is the first-order Doppler effect.

To find ΔE_{ke} , subtract the equations, given in the question, assuming that $\mathbf{k}_{\text{em}} \cdot \mathbf{v}'$ averages to zero. The energy increases by $2E_{\text{rec}}$ for each photon scattered, i.e. each absorption and emission event, as discussed on page 189. For a laser beam in the $-z$ -direction, $\mathbf{k}_{\text{abs}} = -k\hat{\mathbf{e}}_z$, the cooling rate of the energy equals $-\hbar k R_{\text{scatt}} v_z = v_z F_{\text{molasses}}$ as in eqn 9.18.

(9.4) \hbar . Flux of angular momentum: 3×10^{-16} N m.

(9.5) (a) Stopping distances: 7 mm and 1 m.
(b) $f_D/\Delta f_{\text{natural}} = 225$ and 54.

(9.6) 140 μK . 0.36 μK . Ratio: 390

(9.8) (a) Damped harmonic motion.
(b) $z_0 = \hbar k \Gamma / 4\kappa = 2.3 \mu\text{m}$.
(c) $\frac{\partial \alpha}{\partial y} = 0 \Rightarrow y = 1 + x^2$

$\frac{\partial \alpha}{\partial x} = 0 \rightarrow x^2 = 1, y = 2$ and $\alpha_{\text{max}} = \hbar k^2 / 4$. (9.10)

(d) $\tau_{\text{damp}} = 2\hbar / E_{\text{rec}} = 10 \mu\text{s}$.

(9.9) Damping coefficient for atom in two counter-propagating beams:

$$\alpha_{\text{two}} = \hbar k^2 \Gamma \frac{I}{I_{\text{sat}}} \frac{-8(\omega - \omega_0) / \Gamma^2}{\left[1 + 2\frac{I}{I_{\text{sat}}} + \frac{4}{\Gamma^2} (\omega - \omega_0)^2 \right]^2}$$

The atom sees $2I$ not I and this ‘mutual’ saturation is included in the denominator (c.f. eqn 9.4). Write this as

$$\alpha_{\text{two}} = A \frac{-xy}{[1 + y + x^2]^2}$$

where $A = 2\hbar k^2$, $x = 2\delta/\Gamma$ and $y = 2I/I_{\text{sat}}$. Here A and x are the same as in Exercise 9.8 but y is different (by a factor of 2); as in previous Exercise, differentiation w.r.t. x and y shows that maximum occurs for $x = -1$ and $y = 2$ (i.e. $I = I_{\text{sat}}$ in this Exercise) and hence α_{max} is the same as for a single beam.

(b) Damping in the six beam configuration can be written as

$$\alpha_{\text{six}} = \frac{A}{3} \frac{-xy}{[1 + y + x^2]^2}$$

Hence the maximum value is 1/3 of that in previous part (when $I = I_{\text{sat}}/3$): $\alpha_{\text{max}} = \hbar k^2 / 12$. [Assumption of a uniform intensity of $6I$ in the intersection region is a ‘worst-case scenario’; it is not realistic since interference produces a complicated intensity distribution.]

Equation of motion for an atom of mass M :

$$M\ddot{z} + \alpha\dot{z} + \kappa z = 0$$

Critical damping occurs when

$$\frac{\alpha^2}{4M\kappa} = 1$$

Using β defined in eqn 9.31 (and maximum value of α) we find

$$\frac{\alpha^2}{4M\kappa} = \frac{\alpha k}{4M\beta} = \frac{\pi}{12} \frac{E_{\text{rec}}}{\lambda \mu_B \frac{dB}{dz}} = 0.9$$

Almost critical damping for the worst case, hence motion is generally overdamped.

There are two methods:

1. A graphical argument based on Fig. 8.3(a). Any linear function $B(z)$ cannot cross the optimum curve and therefore lies entirely above it on a plot like Fig. 8.3(a). The line cannot have a slope greater than the initial slope of the curve and these conditions allow one to draw the linear function.

2. Alternatively there is a mathematical argument based on the requirement that the deceleration remains less than the maximum that can be produced by the radiation force,

$$v \frac{dv}{dz} < a_{\text{max}} \Rightarrow \frac{dv}{dz} < \frac{a_{\text{max}}}{v}$$

To give some safety margin experiments use $\frac{1}{2}a_{\max}$ or less; this condition is most difficult to fulfil at the the start of the slowing when v has is max. value v_0 .
, than to be consistent we

Chapter 10

(10.1) 140 Hz and 40 Hz.

(10.2) (a) $\omega = 47 \text{ rad s}^{-1}$ ($f = 7 \text{ Hz}$). (b) 5×10^{-8} .
(c) Note error: should be $VT^{3/2}$, or $TV^{2/3}$ is constant (and in footnote 48, $\gamma = 5/3$ for monatomic ideal gas).

(10.3) (a) Hyperfine splitting is $2A = 1772 \text{ MHz}$.
(b) Weak field means Zeeman energy less than hyperfine splitting (strength of hyperfine interaction).
(c) Levels with $J = 1/2$ have $g_F = \pm \frac{gJ}{2I+1}$.
(d) Low-field seeking states are trapped: $F = 1, M_F = -1$; $F = 2, M_F = 1$ and $M_F = 2$.
(e) Ratio = 37 for $F = 2, M_F = 2$, and half this value for the states with $|M_F| = 1$.
(f) Depth (in temperature units) is given by $k_B T_{\max} = g_F \mu_B M_F B_{\max}$ thus $T_{\max} = 10 \text{ mK}$, and half this value for the states with $|M_F| = 1$. (The correction for the effect of gravity is negligible for a light atom such as sodium.)
(g) The r.f. radiation drives transitions at a radial distance from the axis $r = hf/(g_F \mu_B b')$ = 6.7 mm, where $g_F \mu_B/h$ was calculated in part (c), and $f = 70 \text{ MHz}$. This reduces the trap depth to $T' = 0.67 T_{\max}$ where T_{\max} is the value in part (f).

(10.4) Initial Boltzmann distribution:

$$\mathcal{N}(E) = Ae^{-\beta E} \quad \text{where } \beta = k_B T$$

Normalization condition:

$$\mathcal{N}_0 = A \int_0^\infty e^{-\beta E} dE = A/\beta$$

Total energy:

$$E = A \int_0^\infty E e^{-\beta E} dE = A/\beta^2 = \mathcal{N}_0 k_B T$$

Mean energy:

$$\bar{E} = 1/\beta = k_B T$$

For a distribution truncated at energy c :

$$\mathcal{N} = A \int_0^c e^{-\beta E} dE = \mathcal{N}_0 (1 - e^{-\beta c})$$

$$E = A \int_0^c E e^{-\beta E} dE = \frac{A}{\beta^2} (1 - e^{-\beta c} - \beta c e^{-\beta c})$$

Therefore the mean energy after truncation is:

$$\bar{E} = k_B T \left(1 - \frac{\beta c e^{-\beta c}}{1 - e^{-\beta c}} \right)$$

Fractional changes (in terms of $x = c/k_B T$):

(a) $\frac{\Delta \mathcal{N}}{\mathcal{N}} = e^{-x}$, (b) $\frac{\Delta E}{E} = \frac{x e^{-x}}{1 - e^{-x}}$, and (c)

x	$\exp(-x) = \frac{\Delta \mathcal{N}}{\mathcal{N}}$	$\frac{\Delta E}{E}$	$\frac{\Delta E/E}{\Delta \mathcal{N}/\mathcal{N}} \simeq x$
3	$e^{-3} = 5\%$	16%	3.2
6	$e^{-6} = 0.25\%$	1.5%	6.0

Cutting less deeply is more efficient, i.e. it decreases the temperature more for a given loss of atoms. In this example 20 small cuts with $\beta c = 6$ would give about the same loss of atoms as a single cut with $\beta c = 3$, but the many small cuts reduce the energy by twice as much as the single large cut.

(d)

Density $n \sim \mathcal{N}/4r^3$ and $\frac{1}{2}M\omega^2 r^2 = \frac{1}{2}k_B T \propto E$ therefore $n \propto \mathcal{N}/E^{3/2}$. Speed $v \propto E^{1/2}$. So the collision rate

$$R_{\text{coll}} = nv\sigma \propto \frac{\mathcal{N}}{E^{3/2}} E^{1/2} = \frac{\mathcal{N}}{E}$$

Write this as $R = A\mathcal{N}/E$ where $A = \text{constant}$.

$$\Delta R = A \left(\frac{\Delta \mathcal{N}}{E} - \frac{\Delta E}{E^2} \mathcal{N} \right)$$

$$\frac{\Delta R}{R} = \frac{\Delta \mathcal{N}}{\mathcal{N}} - \frac{\Delta E}{E}$$

The collision rate increases, e.g. using the values from part (c), a cut with $x = 3$ gives

$$\frac{\Delta R}{R} = -5\% - (-16\%) = +11\%$$

Both $\Delta\mathcal{N}$ and ΔE are negative. A more general mathematical expression for $\Delta R/R$ can be found from the fractional changes in (a) and (b) above.

Comment: The density of states is taken to be independent of energy (as in a two-dimensional system) to illustrate the principle of evaporative cooling without complicating mathematical details.

- (10.5) Critical temperature $T_C = 0.5 \mu\text{K}$.
Radius of cloud at T_C ,

$$R = \left(\frac{k_B T_C}{M\omega^2} \right)^{1/2} = a_{\text{ho}} N^{1/6} = 11 \mu\text{m}.$$

Number density $n \simeq N/4R^3 = 2 \times 10^{14} \text{cm}^{-3}$.
[Value of $4 \times 10^{13} \text{cm}^{-3}$ for the critical density on p. 228 refers to sodium (this is not made clear).]

- (10.6) (a) Ratio of nonlinear term to kinetic energy is $1.06\mathcal{N}_0 a/b \simeq 5 \times 10^3$, if the size $b = a_{\text{ho}}$; thus neglect k.e. and the actual size is determined by a balance between the nonlinear term and the trapping potential. (b) $b = (1.6\mathcal{N}_0 a/a_{\text{ho}})^{1/5} a_{\text{ho}} = 6 a_{\text{ho}} = 6 \mu\text{m}$. (c) Rough estimate: $n \simeq \mathcal{N}_0/4b^3 \simeq 10^{15} \text{cm}^{-3}$. [Or more carefully, taking density $\propto |\psi|^2$:

$$\begin{aligned} \mathcal{N}_0 &= \int_0^\infty n_{\text{peak}} e^{-r^2/b^2} 4\pi r^2 dr \\ &= n_{\text{peak}} 4\pi b^3 \int_0^\infty x^2 e^{-x^2} dx \end{aligned}$$

which gives the peak density at the centre as $n_{\text{peak}} \simeq \mathcal{N}_0/(\pi^{3/2} b^3) = 7 \times 10^{14} \text{cm}^{-3}$.] (e)

$$\begin{aligned} E &= \frac{3}{4} \hbar\omega \left(\frac{b}{a_{\text{ho}}} \right) \left(1 + \frac{2}{3} \right) \\ &= 44 \hbar\omega \end{aligned}$$

Hence $E/k_B = 0.2 \mu\text{K}$.

(f) When the potential is switched off the repulsive energy, equal to $(2/5)E$, converts into kinetic energy (and the p.e. disappears); the atoms fly outwards with a speed v given by

$$\frac{1}{2} Mv^2 = \frac{2}{5} E$$

This gives $v = 4 \text{mm s}^{-1}$ so that after 30 ms time of flight the cloud has a radius of 120 μm .

(b) $\mu = \mathcal{A}N^{2/5} \Rightarrow E = \frac{5}{7} \mathcal{A}N^{7/5}$; \mathcal{A} constant.

The cloud is cigar-shaped with an aspect ratio $\sqrt{250/16} \simeq 4$. From the Uncertainty principle we estimate the momentum spread along each direction, e.g. $p_x = \hbar/\Delta x$ (neglecting numerical factors of order $1/\sqrt{2}$) where $\Delta x = \sqrt{\hbar/M\omega_r}$ is the size; hence $v_x = \sqrt{\hbar\omega_r/M} = 1.1 \text{mm s}^{-1}$ and similarly along the y -axis—the expansion along the axial direction will be 4 times slower and can be ignored in a rough estimate (as can the initial size in the radial direction). Thus the radial size will become equal to that along the z -axis when $v_x t = \Delta z = \sqrt{\hbar/M\omega_z} = 2.7 \mu\text{m}$ which gives $t \simeq 3 \text{ms}$.

(10.9) $v_s/R = \sqrt{\mu/MR^2}$ and eqn 10.40.

(10.10) Further discussion of dispersion relation in Pethick & Smith (2001), and Pitaevskii & Stringari (2003).

(10.11) Note correction.

$$\frac{4}{3} \frac{E}{\hbar\omega} = x^{-2} + x^2 + Gx^{-3}$$

where $x = b/a_{\text{ho}}$ and $G = 1.06\mathcal{N}a/a_{\text{ho}}$ (c.f. eqn 10.35). For $G = 0$, the minimum occurs at $x = 1$, i.e. $b = a_{\text{ho}}$ as expected for a quantum harmonic oscillator. The condensate shrinks as the strength of the attractive interactions increases until the minimum disappears at $G = -0.36$ where the system collapses (implodes according to this simple model). The maximum number of atoms is only 135.

Chapter 11

(11.2) (a) $1 \times 10^{-10} \text{m}$. (b) $7 \times 10^{-6} \text{m}$ (from eqn 11.8).

(11.3) (a) Correction: $0 \leq u \leq 10\pi/d$. (b) Zero. The second diffracted order occurs at $u = 4\pi/d$, and $ua/2 = \pi$ so that at this position the single-slit diffraction pattern (that contains $\sin(ua/2)$) gives no intensity. (c) For large atoms, the decrease in the effective width of the slits leads to an increase in the angular spread of the single-slit diffraction pattern so

its zero no longer falls exactly on the second order of the diffraction from the grating. Plotting a graph of $(\sin(x)/x)^2$ shows that this function has the value 0.05 at $x = 2.54$. Therefore $u(a - 2r)/2 = 2.54$ where $u = 4\pi/d = 2\pi/a$. This implies that $(a - 2r)/a = 2.54/\pi$, hence $2r = a(1 - 2.54/\pi) = 0.19a$ and $r \simeq 10$ nm.

Chapter 12

(12.1) [Define $e_M^2 \equiv e^2/4\pi\epsilon_0 = 2.3 \times 10^{-28}$ J m.]

(a) 350 kHz.

(c) $a = (2e_M^2/M\omega_z^2)^{2/3} = 11 \mu\text{m}$.

(d) Subtracting the two equations, and dividing through by M gives:

$$\begin{aligned} \ddot{z} + \omega_z^2(a+z) &= \frac{2e_M^2}{M(a+z)^2} = \frac{2e_M^2}{Ma^2} \left(1 + \frac{z}{a}\right)^{-2} \\ &= \frac{2e_M^2}{Ma^2} \left(1 - \frac{2z}{a}\right) \end{aligned}$$

For $z = \dot{z} = 0$ we find the static separation as in part (c), i.e. $\omega_z^2 a = 2e_M^2/Ma^2$. Combining the terms linear in z gives

$$\begin{aligned} \ddot{z} &= -\omega_z^2 \left(1 + \frac{4e_M^2}{Ma^3\omega_z^2}\right) z \\ \ddot{z} &= -3\omega_z^2 z \end{aligned}$$

Oscillation frequency is $\sqrt{3}\omega_z/2\pi = 600$ kHz.

(12.2) (a) $V_{\max} = 0.9 \times M\Omega^2 r_0^2/2e = 3.3 \times 10^4$ V.

(b) $\omega_r = 0.9 \times \Omega/4 \rightarrow f_r = 6.4 \times 10^6$ s⁻¹. For the same trapping conditions a calcium ion has q_x greater than for mercury by the ratio of their atomic masses $200/40 = 5$; hence $q_x = (0.9/\sqrt{2}) \times 5 = 3.2$ and the calcium ion is not confined (eqn 12.20).

(c) Estimate trap depth $U = \frac{1}{2}M\omega_r^2 r_{\max}^2$ by assuming that $r_{\max} \simeq r_0$; using eqn 12.19 gives (13.4) (a)

$$\frac{U}{eV} = \frac{eV}{4M\Omega^2 r_0^2} = \frac{q}{8} = \frac{0.9}{8\sqrt{2}} = 0.08.$$

(12.5) (a) Binding energy of electron in 1s configuration = $14^2 \times 13.6$ eV (neglected repulsion between electrons, see Exercise 3.1); hence voltage = 2.7×10^3 V.

(b) $a_0/14 = 3.8 \times 10^{-12}$ m.

(c) Binding energy = $92^2 \times 13.6 = 0.11$ MeV, which equals $5.2 \times 10^{-7} Mc^2$.

(d) H: 3.2×10^{-14} . U: 1×10^{-8} . It would be possible to measure $\Delta M/M$ for U⁺⁹¹ in a Penning trap (see Section 12.7.2), if the ions could be transferred from a source of highly-ionized atoms into such a trap.

Chapter 13

(13.1) (a) |11⟩. (b) |00⟩ + |11⟩. (c) No. (e) Cannot be written as product, hence entangled.

(13.2) (a)

$$\hat{U}_{\text{CROT}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\hat{U}_{\text{CROT}}^\dagger \hat{U}_{\text{CROT}} = I$$

(b)

$$\hat{U}_{\text{H}}(2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\hat{U}_{\text{H}}^\dagger \hat{U}_{\text{H}} = I$$

(c)

$$\hat{U}_{\text{CNOT}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \hat{U}_{\text{H}}^\dagger(2) \hat{U}_{\text{CROT}} \hat{U}_{\text{H}}(2)$$

$$\hat{U}_{\text{CNOT}} = \hat{U}_{\text{SWAP}}^\dagger(2,3) \hat{U}_{\text{CROT}}(1,2) \hat{U}_{\text{SWAP}}(2,3)$$

(b)

$$|000\rangle \rightarrow (|0\rangle + |1\rangle)(|0\rangle + |1\rangle)(|0\rangle + |1\rangle) = |000\rangle + |001\rangle + |010\rangle \dots$$

(c) $2^{30} \simeq (10^3)^3 \simeq 10^9 = 1$ G qubit [$2^{10} \simeq 10^3$]

(13.4) Both (b) and (c) $|00\rangle + |01\rangle + |10\rangle - |11\rangle$. This state can be written as $00 + 01 + 10 - 11$ without any loss of information and this notation is used this and subsequent answers.

(d) $00 - 01 - 10 - 11$. [(e) answer given.]

(f) 11

(g)

$$\begin{array}{lcl} \psi_{\text{init}} & \xrightarrow{f(x)} & 00 + 01 + 10 - 11 \\ & \xrightarrow{H} & 00 - 01 + 10 + 11 \\ & \xrightarrow{Z} & 00 + 01 - 10 + 11 \\ & \xrightarrow{\text{CROT}} & 00 + 01 - 10 - 11 \\ & \xrightarrow{H} & 10 \end{array}$$

Or more concisely as a table containing the sign of the amplitudes for each state:

	00	01	10	11
initial	+	+	+	+
$f(x)$	+	+	-	+
H	+	-	+	+
U_Z	+	+	-	+
CROT	+	+	-	-
H	+	+	4	+

(13.5) (a) $|\langle 1 | \psi(\tau) \rangle|^2 = \sin^2\left(\frac{\Omega\tau}{2}\right) \simeq \left(\frac{\Omega\tau}{2}\right)^2 = x^2$, where $x = \Omega\tau/2$.

Probability of $|1\rangle$ is $\sin^2\left(\frac{\Omega\tau}{2}\right) \simeq x^2$.

Probability of $|0\rangle$ is $\cos^2\left(\frac{\Omega\tau}{2}\right) \simeq 1 - x^2$.

(b)

Probability of $|1\rangle$ is $p_1(\tau/2) = (x/2)^2 = x^2/4$.

Probability of $|0\rangle$ is $p_0(\tau/2) = 1 - x^2/4$.

(c)

$$\begin{aligned} p_1(\tau) &= p_1\left(\frac{\tau}{2}\right) \cos^2\left(\frac{\Omega\tau}{4}\right) + p_1\left(\frac{\tau}{2}\right) \sin^2\left(\frac{\Omega\tau}{4}\right) \\ &= \frac{x^2}{4} \left(1 - \frac{x^2}{4}\right) + \left(1 - \frac{x^2}{4}\right) \frac{x^2}{4} \\ &= \frac{x^2}{2} \end{aligned}$$

This is half the probability in part (a).

(d) After three measurements:

$$\begin{aligned} p_1(\tau) &= 3 \times \frac{x^2}{9} \left(1 - \frac{x^2}{9}\right) \left(1 - \frac{x^2}{9}\right) \\ &= \frac{x^2}{3} \end{aligned}$$

After n measurements

$$\begin{aligned} p_1(\tau) &= \frac{x^2}{n} \\ p_0(\tau) &= 1 - \frac{x^2}{n} \simeq e^{-x^2/n} \end{aligned}$$

The Fourier theorem sets a limit on the minimum frequency width for a measurement in a finite time τ (c.f. transit-time broadening in eqn 8.10).

(13.6) Initial state:

$$\alpha(a|000\rangle + b|111\rangle) + \beta(a|100\rangle + b|011\rangle) + \gamma(a|010\rangle + b|101\rangle) + \delta(a|001\rangle + b|110\rangle).$$

After CNOT(1,2)

$$\alpha(a|000\rangle + b|101\rangle) + \beta(a|110\rangle + b|011\rangle) + \gamma(a|010\rangle + b|111\rangle) + \delta(a|001\rangle + b|100\rangle).$$

After CNOT(1,3)

$$\begin{aligned} &\alpha(a|000\rangle + b|100\rangle) + \beta(a|111\rangle + b|011\rangle) + \\ &\gamma(a|010\rangle + b|110\rangle) + \delta(a|001\rangle + b|101\rangle) \\ &= (a|0\rangle + b|1\rangle) \{ \alpha|00\rangle + \gamma|10\rangle + \delta|01\rangle \} + \\ &(a|1\rangle + b|0\rangle) \beta|11\rangle. \end{aligned}$$

Measure qubits 2 and 3; if $|11\rangle$ ($\equiv |x11\rangle$ with x arbitrary) then perform NOT operation on qubit 1.